

ALGEBRAIC SUPERGROUPS OF CARTAN TYPE

F. Gavarini

*Dipartimento di Matematica, Università di Roma “Tor Vergata”
via della ricerca scientifica 1 — I-00133 Roma, Italy
e-mail: gavarini@mat.uniroma2.it*

Abstract

¹ I present a construction of connected affine algebraic supergroups \mathbf{G}_V associated with simple Lie superalgebras \mathfrak{g} of Cartan type and with \mathfrak{g} -modules V . Conversely, I prove that every connected affine algebraic supergroup whose tangent Lie superalgebra is of Cartan type is necessarily isomorphic to one of the supergroups \mathbf{G}_V that I introduced. In particular, the supergroup associated in this way with $\mathfrak{g} = W(n)$ and its standard representation is described.

DEDICATED TO PIERRE CARTIER,

WITH GREAT ADMIRATION, ON THE OCCASION OF HIS 80TH BIRTHDAY.

1 Introduction

A real milestone in classical Lie theory is the celebrated classification theorem for complex finite dimensional simple Lie algebras. A similar key result is the classification of all complex finite dimensional simple Lie superalgebras (cf. [14]); in particular, this ensures that these objects form two disjoint families: those of *classical* type, and those of *Cartan* type. The “classical” ones are strict super-analogue of simple, f.d. complex Lie algebras; the “Cartan” ones instead are a super-analogue of complex Lie algebras of Cartan type, which are simple but *infinite* dimensional.

As in the standard Lie context, one can base upon this classification result to tackle the classification problem of existence, construction and uniqueness of simple Lie supergroups, or even simple algebraic supergroups. A super-analogue of Lie’s Third Theorem solves it for Lie supergroups: but the question remains for construction and for the whole algebraic point of view.

In the standard context, a constructive procedure providing all (f.d., connected) simple algebraic groups was provided by Chevalley, over fields; one starts with a (complex) f.d. simple Lie algebra \mathfrak{g} , a faithful \mathfrak{g} -module V , and eventually realizes a group of requested type as a subgroup of $\mathrm{GL}(V)$. In particular, this yields all connected algebraic groups whose tangent Lie algebra is a (f.d.) simple one; this method (and result) also extends to the framework of reductive \mathbb{Z} -group schemes. By analogy, one might try to adapt Chevalley’s method to the f.d. simple Lie superalgebras of classical type, so to provide connected algebraic supergroup-schemes (over \mathbb{Z}) which “integrate” any such Lie superalgebra. This is done in [9] — see also [8] and [11]. In this paper instead I implement Chevalley’s idea to simple Lie superalgebras of Cartan type, with full success: the main result is an existence result, via a constructive procedure, for connected, algebraic supergroup-schemes (over any ring, e.g. \mathbb{Z}) whose tangent Lie superalgebra be simple of Cartan type. As a second result, I prove also a uniqueness theorem for algebraic supergroups of the above mentioned type.

Hereafter I shortly sketch how the present work is organized.

The initial datum is a f.d. simple Lie superalgebra of Cartan type, say \mathfrak{g} . Basing upon a detailed description of the root spaces (with respect to a fixed Cartan subalgebra), I introduce the key notion of *Chevalley basis*. Then I prove two basic results: the existence of Chevalley bases, and a PBW-like theorem for the “Kostant \mathbb{Z} -form” of the universal enveloping superalgebra of \mathfrak{g} .

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Next I take a faithful \mathfrak{g} -module V , and I show that there exists a lattice M in V fixed by the Kostant superalgebra and also by a certain integral form \mathfrak{g}_V of \mathfrak{g} . I define a functor G_V from the category $(\text{salg})_{\mathbb{k}}$ of commutative \mathbb{k} -superalgebras to the category (groups) of groups as follows: for $A \in (\text{salg})_{\mathbb{k}}$, I let $G_V(A)$ be the subgroup of $\text{GL}(A \otimes_{\mathbb{Z}} M)$ generated by “homogeneous one-parameter subgroups” associated with the root vectors and with the toral elements in a Chevalley basis. Then I pick the sheafification \mathbf{G}_V (in the sense of category theory) of the functor G_V .

Using commutation relations among generators, I find a factorization of \mathbf{G}_V into direct product of representable (algebraic) superschemes: thus \mathbf{G}_V itself is representable, hence it is an “affine (algebraic) supergroup”. Some extra work shows how \mathbf{G}_V depends on V , that it is independent of the choice of M and that its tangent Lie superalgebra is \mathfrak{g}_V . So the construction of \mathbf{G}_V yields an *existence theorem* of a supergroup having \mathfrak{g}_V as tangent Lie superalgebra. Right after, I prove the converse, i.e. a *uniqueness theorem* showing that any such supergroup is isomorphic to some \mathbf{G}_V .

Finally, I illustrate the example of \mathbf{G}_V for \mathfrak{g} of type $W(n)$ and V its defining representation — i.e. the Grassmann algebra in n odd indeterminates, $W(n)$ being the algebra of its superderivations.

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2 Preliminaries

We introduce hereafter some preliminaries of supergeometry (main references are [4], [15], [18]).

2.1 Superalgebras, superspaces, supergroups

Let \mathbb{k} be a unital, commutative ring. We call \mathbb{k} -*superalgebra* any associative, unital \mathbb{k} -algebra A which is \mathbb{Z}_2 -graded, where \mathbb{Z}_2 is the two-element group $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$: thus $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $A_{\bar{a}} A_{\bar{b}} \subseteq A_{\bar{a}+\bar{b}}$. The \mathbb{k} -submodule $A_{\bar{0}}$ and its elements are called *even*, $A_{\bar{1}}$ and its elements *odd*. By $p(x) \in \mathbb{Z}_2$ we denote the *parity* of any homogeneous element $x \in A_{p(x)}$. All \mathbb{k} -superalgebras form a category, whose morphisms are those in the category of algebras preserving the unit and the \mathbb{Z}_2 -grading. For any $n \in \mathbb{N}$ we call $A_{\bar{1}}^n$ the $A_{\bar{0}}$ -span in A of all products $\vartheta_1 \cdots \vartheta_n$ with $\vartheta_i \in A_{\bar{1}}$ for all i , and $A_{\bar{1}}^{(n)}$ the unital subalgebra of A generated by $A_{\bar{1}}^n$. A superalgebra A is *commutative* iff $xy = (-1)^{p(x)p(y)}yx$ for all homogeneous $x, y \in A$ and $z^2 = 0$ for all odd $z \in A_{\bar{1}}$. We denote by $(\text{salg})_{\mathbb{k}}$ the category of commutative \mathbb{k} -superalgebras, dropping the subscript \mathbb{k} if unnecessary.

Definition 2.1. A *superspace* $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$ with a sheaf of commutative superalgebras \mathcal{O}_S such that the stalk $\mathcal{O}_{S,x}$ is a local superalgebra for all $x \in |S|$. A *morphism* $\phi : S \rightarrow T$ of superspaces consists of a pair $\phi = (|\phi|, \phi^*)$, where $|\phi| : |S| \rightarrow |T|$ is a morphism of topological spaces and $\phi^* : \mathcal{O}_T \rightarrow \phi_* \mathcal{O}_S$ is a sheaf morphism such that $\phi_x^*(\mathfrak{m}_{|\phi|(x)}) \subseteq \mathfrak{m}_x$ where $\mathfrak{m}_{|\phi|(x)}$ and \mathfrak{m}_x are the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$, and ϕ_x^* is the morphism induced by ϕ^* on the stalk. Here as usual $\phi_* \mathcal{O}_S$ is the direct image (on $|T|$) of $\mathcal{O}_S(V)$.

Given a superspace $S = (|S|, \mathcal{O}_S)$, let $\mathcal{O}_{S,\bar{0}}$ and $\mathcal{O}_{S,\bar{1}}$ be the sheaves on $|S|$ defined as follows: $\mathcal{O}_{S,\bar{0}}(U) := \mathcal{O}_S(U)_{\bar{0}}$, $\mathcal{O}_{S,\bar{1}}(U) := \mathcal{O}_S(U)_{\bar{1}}$ for each open subset U in $|S|$. Then $\mathcal{O}_{S,\bar{0}}$ is a sheaf of ordinary commutative algebras, while $\mathcal{O}_{S,\bar{1}}$ is a sheaf of $\mathcal{O}_{S,\bar{0}}$ -modules.

Definition 2.2. A *superscheme* is a superspace $S := (|S|, \mathcal{O}_S)$ such that $(|S|, \mathcal{O}_{S,\bar{0}})$ is an ordinary scheme and $\mathcal{O}_{S,\bar{1}}$ is a quasi-coherent sheaf of $\mathcal{O}_{S,\bar{0}}$ -modules. A *morphism* of superschemes is one of the underlying superspaces. The (*super*)*dimension* of S is by definition the pair $\dim(|S|) \mid rk(\mathcal{O}_{S,\bar{1}})$ where $rk(\mathcal{O}_{S,\bar{1}})$ is the rank of the quasi-coherent sheaf of $\mathcal{O}_{S,\bar{0}}$ -modules $\mathcal{O}_{S,\bar{1}}$.

Definition 2.3. Let $A \in (\text{salg})_{\mathbb{k}}$ and let $\mathcal{O}_{A_{\bar{0}}}$ be the structural sheaf of the ordinary scheme $\text{Spec}(A_{\bar{0}}) = (\text{Spec}(A_{\bar{0}}), \mathcal{O}_{A_{\bar{0}}})$, where $\text{Spec}(A_{\bar{0}})$ denotes the prime spectrum of $A_{\bar{0}}$. Now A is an $A_{\bar{0}}$ -module, so we have a sheaf \mathcal{O}_A of $\mathcal{O}_{A_{\bar{0}}}$ -modules over $\text{Spec}(A_{\bar{0}})$ with stalk A_p , the p -localization of the $A_{\bar{0}}$ -module A , at any $p \in \text{Spec}(A_{\bar{0}})$. We set $\underline{\text{Spec}}(A) := (\text{Spec}(A_{\bar{0}}), \mathcal{O}_A)$: by definition,

this is a superscheme. We call *affine* any superscheme which is isomorphic to $\underline{\text{Spec}}(A)$ for some $A \in (\text{salg})_{\mathbb{k}}$; any affine superscheme is *algebraic* if its representing superalgebra is finitely generated.

Clearly any superscheme is locally isomorphic to an affine superscheme.

Example 2.4. We call *affine superspace* the superscheme $\mathbb{A}_{\mathbb{k}}^{p|q} := \underline{\text{Spec}}(\mathbb{k}[x_1, \dots, x_p] \otimes_{\mathbb{k}} \mathbb{k}[\xi_1 \dots \xi_q])$ ($p, q \in \mathbb{N}$), also denoted $\mathbb{k}^{p|q}$: here $\mathbb{k}[\xi_1 \dots \xi_q]$ is the exterior algebra generated by q anticommuting indeterminates, and $\mathbb{k}[x_1, \dots, x_p]$ the polynomial algebra in p commuting indeterminates. \diamond

Definition 2.5. Let X be a superscheme. Its *functor of points* is the functor $h_X : (\text{salg})_{\mathbb{k}} \rightarrow (\text{sets})$ defined on objects by $h_X(A) := \text{Hom}(\underline{\text{Spec}}(A), X)$ and on arrows by $h_X(f)(\phi) := \phi \circ \underline{\text{Spec}}(f)$. When h_X is actually a functor from $(\text{salg})_{\mathbb{k}}$ to (groups), the category of groups, we say that X is a *supergroup-scheme*. If X is affine, this is equivalent to the fact that $\mathcal{O}(X)$ — the superalgebra of global sections of the structure sheaf on X — is a (commutative) *Hopf superalgebra*. More in general, we shall call *supergroup functor* any functor $G : (\text{salg})_{\mathbb{k}} \rightarrow (\text{groups})$.

Any representable supergroup functor is the same as an affine supergroup: indeed, the former corresponds to the functor of points of the latter. See [3], Ch. 3–5, for more details.

In the present work we consider only affine supergroups, described via their functor of points: we introduce them as supergroup functors, and then show that they are representable and algebraic.

Examples 2.6.

(a) Let V be a free \mathbb{k} -supermodule. For any commutative \mathbb{k} -superalgebra A we define $V(A) := (A \otimes_{\mathbb{k}} V)_{\bar{0}} = A_{\bar{0}} \otimes_{\mathbb{k}} V_{\bar{0}} \oplus A_{\bar{1}} \otimes_{\mathbb{k}} V_{\bar{1}}$. When V is finite dimensional, this is a representable functor (from $(\text{salg})_{\mathbb{k}}$ to \mathbb{k} -super-vector spaces). Hence V can be seen as an affine superscheme.

(b) *GL(V) as an affine algebraic supergroup.* Let V be a free \mathbb{k} -supermodule of finite (super-)rank $p|q$. For any superalgebra A , let $\text{GL}(V)(A) := \text{GL}(V(A))$ be the set of isomorphisms $V(A) \rightarrow V(A)$ preserving the \mathbb{Z}_2 -grading. If we fix a homogeneous basis for V , we see that $V \cong \mathbb{k}^{p|q}$; in other words, $V_{\bar{0}} \cong \mathbb{k}^p$ and $V_{\bar{1}} \cong \mathbb{k}^q$. In this case, we also denote $\text{GL}(V)$ with $\text{GL}_{p|q}$. Now, $\text{GL}_{p|q}(A)$ is the group of invertible matrices of size $(p+q)$ with diagonal block entries in $A_{\bar{0}}$ and off-diagonal block entries in $A_{\bar{1}}$. It is known that the functor $\text{GL}(V)$ is representable, so $\text{GL}(V)$ is indeed an affine supergroup, and also algebraic; see (e.g.), [18], Ch. 3, for further details.

Note that every element $\text{GL}(V(A))$ extends to a (degree-preserving, A -linear) automorphism of $V_A := A \otimes_{\mathbb{k}} V$; viceversa, any automorphism of V_A restricts to an element of $\text{GL}(V(A))$. So $\text{GL}(V(A))$ identifies with $\text{GL}(V_A)$, the group of (A -linear) automorphisms of V_A . We call $\text{GL}(V_{\bullet})$ the obvious functor from $(\text{salg})_{\mathbb{k}}$ to (groups) given on objects by $A \mapsto \text{GL}(V_{\bullet})(A) := \text{GL}(V_A)$. \diamond

2.2 Lie superalgebras

The notion of Lie superalgebra over a field is well known, at least for characteristic neither 2 nor 3. To take into account all cases, we consider the following modified formulation: it is a “correct” notion of Lie superalgebras given by the standard notion enriched with an additional “2-mapping”, a close analogue to the p -mapping in a p -restricted Lie algebra over a field of characteristic $p > 0$.

Definition 2.7. (cf. [1], [7]) Let $A \in (\text{salg})_{\mathbb{k}}$. We call *Lie A -superalgebra* any A -supermodule $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ endowed with a (*Lie super*)*bracket* $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, $(x, y) \mapsto [x, y]$, and a *2-operation* $(\cdot)^{(2)} : \mathfrak{g}_{\bar{1}} \rightarrow \mathfrak{g}_{\bar{0}}$, $z \mapsto z^{(2)}$, such that (for all $x, y \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}$, $w \in \mathfrak{g}_{\bar{0}}$, $z, z_1, z_2 \in \mathfrak{g}_{\bar{1}}$):

- (a) $[\cdot, \cdot]$ is A -superbilinear (in the obvious sense), $[w, w] = 0$, $[z[z, z]] = 0$;
- (b) $[x, y] + (-1)^{p(x)p(y)}[y, x] = 0$ (*anti-symmetry*);
- (c) $(-1)^{p(x)p(z)}[x, [y, z]] + (-1)^{p(y)p(x)}[y, [z, x]] + (-1)^{p(z)p(y)}[z, [x, y]] = 0$ (*Jacobi identity*);
- (d) $(\cdot)^{(2)}$ is A -quadratic, i.e. $(a_{\bar{0}} z)^{(2)} = a_{\bar{0}}^2 z^{(2)}$, $(a_{\bar{1}} w)^{(2)} = 0$ for $a_{\bar{0}} \in A_{\bar{0}}$, $a_{\bar{1}} \in A_{\bar{1}}$;
- (e) $(z_1 + z_2)^{(2)} = z_1^{(2)} + [z_1, z_2] + z_2^{(2)}$, $[z^{(2)}, x] = [z, [z, x]]$.

All Lie A -superalgebras form a category, whose morphisms are the A -superlinear (in the obvious sense), graded maps preserving the bracket and the 2-operation.

A Lie superalgebra is said to be *simple* if it has no non-trivial homogenous ideal. Simple Lie superalgebras of finite dimension over algebraically closed fields of characteristic zero were classified by V. Kac (cf. [14]), to whom we shall refer for the standard terminology and notions.

Examples 2.8. (a) Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be any associative \mathbb{k} -superalgebra. There is a canonical structure of Lie superalgebra on \mathcal{A} given by $[x, y] := xy - (-1)^{p(x)p(y)}yx$ for all homogeneous $x, y \in \mathcal{A}_0 \cup \mathcal{A}_1$ and 2-operation $z^{(2)} := z^2 = zz$ (the associative square in \mathcal{A}) for all odd $z \in \mathcal{A}_1$.

(b) Let $V = V_0 \oplus V_1$ be a free \mathbb{k} -supermodule, and consider $End(V)$, the endomorphisms of V as an ordinary \mathbb{k} -module. This is again a free super \mathbb{k} -module, $End(V) = End(V)_0 \oplus End(V)_1$, where $End(V)_0$ are the morphisms which preserve the parity, while $End(V)_1$ are the morphisms which reverse the parity. By the recipe in (a), $End(V)$ is a Lie \mathbb{k} -superalgebra with $[A, B] := AB - (-1)^{p(A)p(B)}BA$, $C^{(2)} := C^2$, for all $A, B, C \in End(V)$ homogeneous, with C odd.

The standard example is for V of finite rank, say $V := \mathbb{k}^{p|q} = \mathbb{k}^p \oplus \mathbb{k}^q$, with $V_0 := \mathbb{k}^p$ and $V_1 := \mathbb{k}^q$: in this case we also write $End(\mathbb{k}^{p|q}) := End(V)$ or $\mathfrak{gl}_{p|q} := End(V)$. Choosing a basis for V of homogeneous elements (writing first the even ones), we identify $End(V)_0$ with the set of all diagonal block matrices, and $End(V)_1$ with the set of all off-diagonal block matrices. \diamond

2.9. Lie superalgebras and Lie algebra valued functors. Let us fix \mathbb{k} and $(\text{salg})_{\mathbb{k}}$ as in section 2.1, and let $(\text{mod})_{\mathbb{k}}$ and $(\text{Lie})_{\mathbb{k}}$ be the category of \mathbb{k} -modules and of Lie \mathbb{k} -algebras. Any \mathbb{k} -supermodule \mathfrak{m} yields a well-defined functor $\mathcal{M}_{\mathfrak{m}} : (\text{salg})_{\mathbb{k}} \rightarrow (\text{mod})_{\mathbb{k}}$, given on objects by $\mathcal{M}_{\mathfrak{m}}(A) := (A \otimes \mathfrak{m})_0 = A_0 \otimes \mathfrak{m}_0 \oplus A_1 \otimes \mathfrak{m}_1$, for all $A \in (\text{salg})_{\mathbb{k}}$. If in addition $\mathfrak{m} = \mathfrak{g}$ is a Lie \mathbb{k} -superalgebra, then $A \otimes \mathfrak{g}$ is a Lie A -superalgebra, its Lie bracket being defined via sign rules by $[a \otimes X, a' \otimes X'] := (-1)^{|X||a'|} a a' \otimes [X, X']$, and similarly for the 2-operation: then $\mathcal{L}_{\mathfrak{g}}(A)$ is its even part, so it is a Lie algebra. Thus we have a Lie algebra valued functor $\mathcal{L}_{\mathfrak{g}} : (\text{salg})_{\mathbb{k}} \rightarrow (\text{Lie})_{\mathbb{k}}$ (see [3], §11.2, for details). We shall call *quasi-representable* any functor $\mathcal{L} : (\text{salg})_{\mathbb{k}} \rightarrow (\text{Lie})_{\mathbb{k}}$ for which there exists a Lie \mathbb{k} -superalgebra \mathfrak{g} such that $\mathcal{L} = \mathcal{L}_{\mathfrak{g}}$: indeed, any such functor is even representable as soon as the \mathbb{k} -module \mathfrak{g} is free of finite rank, since $\mathcal{L}_{\mathfrak{g}}$ is then represented by the commutative \mathbb{k} -superalgebra $S(\mathfrak{g}^*) \in (\text{salg})_{\mathbb{k}}$. In particular, when V is a free super \mathbb{k} -module we have the Lie superalgebra $\mathfrak{g} := End(V)$ and the functor $\mathcal{L}_{End(V)}$; then $GL(V)$ — cf. Example 2.6(b) — is a subfunctor of $\mathcal{L}_{End(V)}$ — as a set-valued functor.

This “functorial presentation” of Lie superalgebras can be adapted to representations too: if \mathfrak{g} is a Lie \mathbb{k} -superalgebra and V a \mathfrak{g} -module, the representation map $\phi : \mathfrak{g} \rightarrow End(V)$ clearly induces a natural transformation of functors $\mathcal{L}_{\mathfrak{g}} \rightarrow \mathcal{L}_{End(V)}$.

2.3 Lie superalgebras of Cartan type

In the following, \mathbb{K} is an algebraically closed field of characteristic zero.

By definition, a Lie superalgebra \mathfrak{g} over \mathbb{K} is of Cartan type if it is finite dimensional, simple, with the odd part \mathfrak{g}_1 which is not semisimple as a module over the even part \mathfrak{g}_0 . Actually, Cartan type Lie superalgebras split into four countable families, denoted $W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$ with $n \geq 2$, $n \geq 3$, $n \geq 4$ (with n being even) and $n \geq 4$ respectively.

We shall now describe in short all these types. For further details, see [14], §3.

Given $n \in \mathbb{N}_+$, denote by $\Lambda(n) = \mathbb{K}[\xi_1, \dots, \xi_n]$ the free commutative superalgebra over \mathbb{K} with n odd generators ξ_1, \dots, ξ_n ; this is (isomorphic to) the Grassmann algebra of rank n , and is naturally \mathbb{Z} -graded, with $deg(\xi_i) = 1$. A \mathbb{K} -basis of $\Lambda(n)$ is the set $B_{\Lambda(n)} := \{ \underline{\xi}^{\underline{e}} \mid \underline{e} \in \{0, 1\}^n \}$, where $\underline{\xi}^{\underline{e}} := \xi_1^{\wedge e(1)} \wedge \xi_2^{\wedge e(2)} \wedge \dots \wedge \xi_n^{\wedge e(n)} = \xi_1^{e(1)} \cdot \xi_2^{e(2)} \cdot \dots \cdot \xi_n^{e(n)}$ (hereafter we shall drop the \wedge 's).

For later use, for every $\underline{e} \in \{0, 1\}^n$ we define $|\underline{e}| := \sum_{k=1}^n e(k)$.

2.10. Definition of $W(n)$. For any $n \in \mathbb{N}_+$ with $n \geq 2$, let $W(n) := Der_{\mathbb{K}}(\Lambda(n))$ denote the set of \mathbb{K} -(super)derivations of $\Lambda(n)$. This is a Lie subsuperalgebra of $End_{\mathbb{K}}(\Lambda(n))$: explicitly, one has

$$W(n) = \left\{ \sum_{i=1}^n P_i(\underline{\xi}) \partial_i \mid P_i(\underline{\xi}) \in \Lambda(n) \quad \forall i = 1, \dots, n \right\}$$

where each ∂_i is the unique superderivation such that $\partial_i(\xi_j) = \delta_{i,j}$. This Lie superalgebra $W(n)$ is naturally \mathbb{Z} -graded, with $deg(\partial_i) = -1$, $deg(\xi_i) = +1$; in detail,

$$W(n) = \bigoplus_{z \in \mathbb{Z}} W(n)_z, \quad W(n)_z = \left\{ \sum_{i=1}^n P_i(\underline{\xi}) \partial_i \mid P_i(\underline{\xi}) \in \Lambda(n)_{z+1} \ \forall i \right\} \quad (2.1)$$

so $W(n)_z \neq \{0\}$ iff $-1 \leq z \leq n-1$. Thus, if $W(n)_{[z]} := \bigoplus_{\zeta \equiv z \pmod{n+1}} W(n)_\zeta$ for all $[z] \in \mathbb{Z}_{n+1}$ (the group of integers modulo $n+1$) the \mathbb{Z} -grading above yields the \mathbb{Z}_{n+1} -grading $W(n) = \bigoplus_{[z] \in \mathbb{Z}_{n+1}} W(n)_{[z]}$.

The \mathbb{Z} -grading yields a \mathbb{Z} -filtration $(W(n) \supseteq) \cdots \supseteq W(n)_{\geq z-1} \supseteq W(n)_{\geq z} \supseteq W(n)_{\geq z+1} \supseteq \cdots$ of $W(n)$ as a Lie superalgebra, where $W(n)_{\geq z} := \bigoplus_{k \geq z} W(n)_k$ for all $z \in \mathbb{Z}$; the associated graded Lie superalgebra then is graded-isomorphic to $W(n)$ itself. Also, this \mathbb{Z} -grading is consistent with the \mathbb{Z}_2 -grading of $W(n)$, i.e. the \mathbb{Z}_2 -grading is given by $W(n) = W(n)_{\bar{0}} \oplus W(n)_{\bar{1}}$ with

$$W(n)_{\bar{0}} := \bigoplus_{z \in 2\mathbb{Z}} W(n)_z, \quad W(n)_{\bar{1}} := \bigoplus_{z \in (2\mathbb{Z}+1)} W(n)_z$$

In particular, one also has the following three facts:

- (a) for each $z \in \mathbb{Z}$, the set $B_{W(n);z} := \{ \underline{\xi}^{\underline{e}} \partial_i \mid \underline{e} \in \{0,1\}^n, i=1,\dots,n; |\underline{e}| = z+1 \}$ is a \mathbb{K} -basis of $W(n)_z$; for each $\bar{z} \in \mathbb{Z}_2$ the set $B_{W(n);\bar{z}} := \bigcup_{(z \bmod 2) = \bar{z}} B_{W(n);z}$ is a \mathbb{K} -basis of $W(n)_{\bar{z}}$;
- the set $B_{W(n)} := \bigcup_{z \in \mathbb{Z}} B_{W(n);z}$ is a — \mathbb{Z} -homogeneous and \mathbb{Z}_2 -homogeneous — \mathbb{K} -basis of $W(n)$.
- (b) $W(n)_{\bar{0}}$ is a Lie subalgebra of the even part $W(n)_{\bar{0}}$ of $W(n)$, isomorphic to $\mathfrak{gl}(n)$, via $\xi_i \partial_j \mapsto e_{i,j}$ (= the elementary $(n \times n)$ -matrix bearing 1 in position (i,j) and zero elsewhere);
- (c) $W(n)_{\bar{1}}$, as a module for $W(n)_{\bar{0}} \cong \mathfrak{gl}(n)$, is the dual of the standard module of $\mathfrak{gl}(n)$.

2.11. The Lie structure in $W(n)$. Our $W(n) := \text{Der}_{\mathbb{K}}(W(n))$ is a Lie subsuperalgebra of $\text{End}_{\mathbb{K}}(\Lambda(n))$, whose Lie bracket in the latter is the “supercommutator” (cf. Example 2.8(b)). Thus we first consider the composition product of two basis elements in $W(n)$. Calculations give

$$\underline{\xi}^{\underline{a}} \partial_j \circ \underline{\xi}^{\underline{b}} \partial_\ell = (-1)^{|\underline{b}|} \underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}} \partial_j \partial_\ell + (-1)^{\#\{s \mid s < j, \underline{b}(s)=1\}} \delta_{\underline{b}(j),1} \underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}-\underline{e}_j} \partial_\ell \quad (2.2)$$

where $\underline{e}_j \in \{0,1\}^n$ is given by $\underline{e}_j(k) := \delta_{j,k}$, and $\underline{b}-\underline{e}_j$ is the obvious element in the set $\{0,1\}^n$. Therefore the defining formula $\left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_\ell \right] = \underline{\xi}^{\underline{a}} \partial_j \circ \underline{\xi}^{\underline{b}} \partial_\ell - (-1)^{\deg(\underline{\xi}^{\underline{a}} \partial_j) \deg(\underline{\xi}^{\underline{b}} \partial_\ell)} \underline{\xi}^{\underline{b}} \partial_\ell \circ \underline{\xi}^{\underline{a}} \partial_j$ (taken from Example 2.8(a)) along with (2.2) yields, taking into account that $\partial_\ell \partial_j = -\partial_j \partial_\ell$,

$$\left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_\ell \right] = \pm \delta_{\underline{b}(j),1} \underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}-\underline{e}_j} \partial_\ell \pm \delta_{\underline{a}(\ell),1} \underline{\xi}^{\underline{b}} \underline{\xi}^{\underline{a}-\underline{e}_\ell} \partial_j \quad (2.3)$$

for all $\underline{a}, \underline{b} \in \{0,1\}^n$, $j, \ell = 1, \dots, n$. In particular — reordering the various factors ξ_k — this shows that $\left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_\ell \right]$ has coefficients in $\{-1, 0, 1\}$ with respect to the basis $B_{W(n)}$.

When $\delta_{\underline{b}(j),1} = 1 = \delta_{\underline{a}(\ell),1}$, i.e. in the case $\underline{b}(j) = 1 = \underline{a}(\ell)$, formula (2.3) looks more precise:

$$\left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_\ell \right] = (-1)^N \underline{\xi}^{\underline{a}-\underline{e}_\ell} \underline{\xi}^{\underline{b}-\underline{e}_j} (\xi_\ell \partial_\ell - \xi_j \partial_j), \quad \forall \underline{a}, \underline{b} : \underline{a}(\ell) = 1 = \underline{b}(j) \quad (2.4)$$

Note also that (2.4) takes a special form in the following three cases:

$$\text{— if } j = \ell, \text{ then } \left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_\ell \right] = 0; \quad (2.5)$$

$$\text{— if } j \neq \ell, \underline{a}(j) = 1, \text{ then } \left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_\ell \right] = (-1)^N \underline{\xi}^{\underline{a}-\underline{e}_\ell} \underline{\xi}^{\underline{b}-\underline{e}_j} \xi_\ell \partial_\ell; \quad (2.6)$$

$$\text{— if } j \neq \ell, \underline{b}(\ell) = 1, \text{ then } \left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_\ell \right] = (-1)^{N+1} \underline{\xi}^{\underline{a}-\underline{e}_\ell} \underline{\xi}^{\underline{b}-\underline{e}_j} \xi_j \partial_j. \quad (2.7)$$

Finally, for the 2-operation the defining formula (from Example 2.8(b)) along with (2.2) gives

$$(\underline{\xi}^{\underline{a}} \partial_j)^{(2)} = (\underline{\xi}^{\underline{a}} \partial_j)^2 = 0 \quad \forall \underline{a} : |\underline{a}| \in 2\mathbb{N} \quad (2.8)$$

2.12. Definition of $S(n)$. We retain notations of Definition 2.10 above, in particular $\Lambda(n)$ and $W(n) := \text{Der}_{\mathbb{K}}(\Lambda(n))$, for $n \in \mathbb{N}_+$, are defined as therein; in addition, we assume now $n \geq 3$.

Define the *divergence operator* $\text{div} : W(n) := \text{Der}_{\mathbb{K}}(\Lambda(n)) \rightarrow \Lambda(n)$ by $\text{div}(\sum_{i=1}^n P_i(\underline{\xi}) \partial_i) := \sum_{i=1}^n \partial_i(P_i(\underline{\xi}))$ for $\sum_{i=1}^n P_i(\underline{\xi}) \partial_i \in W(n)$. Then set

$$S(n) := \left\{ D := \sum_{i=1}^n P_i(\underline{\xi}) \partial_i \in W(n) \mid \text{div}(D) = 0 \right\} \quad (= \text{Ker}(\text{div}))$$

This is a \mathbb{Z} -graded Lie subsuperalgebra of $W(n)$ — with \mathbb{Z} -grading induced from $W(n)$: so

$$S(n) = \bigoplus_{z \in \mathbb{Z}} S(n)_z, \quad S(n)_z = W(n)_z \cap S(n) \quad (2.9)$$

— cf. (2.1) — where $S(n)_z \neq \{0\}$ if and only if $-1 \leq z \leq n-2$ — see below for more details. Like for $W(n)$, this \mathbb{Z} -grading yields also a \mathbb{Z}_n -grading $S(n) = \bigoplus_{[z] \in \mathbb{Z}_n} S(n)_{[z]}$ with

$S(n)_{[z]} := \bigoplus_{\zeta \equiv z \pmod n} S(n)_{\zeta}$ for all $[z] \in \mathbb{Z}_n$ (the group of integers modulo n). Again, the \mathbb{Z} -grading yields a \mathbb{Z} -filtration, which coincides with the one induced by $W(n)$, whose associated graded Lie superalgebra is graded-isomorphic to $S(n)$ itself. Moreover, the \mathbb{Z} -grading of $S(n)$ is consistent with the \mathbb{Z}_2 -grading, in the obvious sense (like for $W(n)$).

The construction and the results for $W(n)$ — cf. Definition 2.10 — give:

(a) a basis of the \mathbb{K} -vector space $S(n)$ is given by

$$B_{S(n)} := \left\{ \underline{\xi}^{\underline{e}} \partial_i \mid \underline{e}(i) = 0 \right\} \cup \left\{ \underline{\xi}^{\underline{e}} (\xi_j \partial_j - \xi_{j'} \partial_{j'}) \mid \begin{array}{l} 1 \leq j < j' \leq n, \underline{e}(j) = 0 = \underline{e}(j') \\ \underline{e}(j'') = 1 \quad \forall j < j'' < j' \end{array} \right\}$$

In the following we call “*of first type*” the elements of this basis of the form $\underline{\xi}^{\underline{e}} \partial_i$, and “*of second type*” those of the form $\underline{\xi}^{\underline{e}} (\xi_j \partial_j - \xi_{j'} \partial_{j'})$; more in general, we call “*(elements) of second type*” also all those of the form $\underline{\xi}^{\underline{e}} (\xi_j \partial_j - \xi_k \partial_k)$, for any $j < k$ with $\underline{e}(j) = 0 = \underline{e}(k)$.

Again, this basis is homogeneous (for both the \mathbb{Z} -grading and the \mathbb{Z}_2 -grading), i.e. $B_{S(n)} = \bigcup_{z \in \mathbb{Z}} B_{S(n);z}$ and $B_{S(n)} = \bigcup_{\bar{z} \in \mathbb{Z}_2} B_{S(n);\bar{z}}$ where $B_{S(n);z} := B_{S(n)} \cap S(n)_z$, respectively $B_{S(n);\bar{z}} := B_{S(n)} \cap S(n)_{\bar{z}}$, is a basis of $S(n)_z$, respectively of $S(n)_{\bar{z}}$, for every $z \in \mathbb{Z}$, $\bar{z} \in \mathbb{Z}_2$.

(b) $S(n)_0$ is a Lie subalgebra of the even part $S(n)_{\bar{0}}$ of $S(n)$, isomorphic to $\mathfrak{sl}(n)$, via $\xi_i \partial_j \mapsto e_{i,j}$ (notation of Definition 2.10) for $i \neq j$, and $(\xi_k \partial_k - \xi_{\ell} \partial_{\ell}) \mapsto (e_{k,k} - e_{\ell,\ell})$ for $k \neq \ell$;

(c) $S(n)_{-1}$, as a module for $S(n)_0 \cong \mathfrak{sl}(n)$, is the dual of the standard module of $\mathfrak{sl}(n)$.

2.13. The Lie structure in $S(n)$. We need formulas for the Lie bracket of elements in $B_{S(n)}$.

First we look at pairs of basis elements of the first type (cf. §2.12, so $\underline{a}(j) = 0$, $\underline{b}(\ell) = 0$). For their bracket, formulas (2.2–7) give (with the right-hand side which in third case might be zero)

$$\left[\underline{\xi}^{\underline{a}} \partial_j, \underline{\xi}^{\underline{b}} \partial_{\ell} \right] = \begin{cases} \pm \underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}-\underline{e}_j} \partial_{\ell} & \text{if } \underline{a}(\ell) = 0, \underline{b}(j) = 1 \\ \pm \underline{\xi}^{\underline{b}} \underline{\xi}^{\underline{a}-\underline{e}_{\ell}} \partial_j & \text{if } \underline{a}(\ell) = 1, \underline{b}(j) = 0 \\ \pm \underline{\xi}^{\underline{a}-\underline{e}_{\ell}} \underline{\xi}^{\underline{b}-\underline{e}_j} (\xi_{\ell} \partial_{\ell} - \xi_j \partial_j) & \text{if } \underline{a}(\ell) = 1, \underline{b}(j) = 1 \\ 0 & \text{if } \underline{a}(\ell) = 0, \underline{b}(j) = 0 \end{cases}$$

Now we consider the Lie bracket of a basis element of first type and one of second type (see §2.12, thus $\underline{e}(\ell) = 0$). Then $\left[\underline{\xi}^{\underline{e}} \partial_{\ell}, \underline{\xi}^{\underline{e}'} (\xi_j \partial_j - \xi_k \partial_k) \right] = \left((-1)^{N'} - (-1)^{N''} \right) \underline{\xi}^{\underline{e}'-\underline{e}_{\ell}} \underline{\xi}^{\underline{e}} \partial_{\ell}$, by formulas (2.2–7), for some $N', N'' \in \mathbb{N}$; a detailed (yet elementary) analysis of signs shows that

$$\left[\underline{\xi}^{\underline{e}} \partial_{\ell}, \underline{\xi}^{\underline{e}'} (\xi_j \partial_j - \xi_k \partial_k) \right] = 0$$

Third, we look at pairs of elements both of the second type: formulas (2.2–7) eventually give

$$\begin{aligned} \left[\underline{\xi}^{\underline{a}} (\xi_j \partial_j - \xi_k \partial_k), \underline{\xi}^{\underline{b}} (\xi_{\ell} \partial_{\ell} - \xi_t \partial_t) \right] &= \\ &= (\underline{b}(j) - \underline{b}(k)) \underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}} (\xi_{\ell} \partial_{\ell} - \xi_t \partial_t) - (\underline{a}(\ell) - \underline{a}(t)) \underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}} (\xi_j \partial_j - \xi_k \partial_k) \end{aligned}$$

We must stress two facts. First, $\underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}} (\xi_{\ell} \partial_{\ell} - \xi_t \partial_t)$ and $\underline{\xi}^{\underline{a}} \underline{\xi}^{\underline{b}} (\xi_j \partial_j - \xi_k \partial_k)$ are both either zero or elements of $B_{S(n)}$ of the second type; second, $(\underline{b}(j) - \underline{b}(k)), (\underline{a}(\ell) - \underline{a}(t)) \in \{-1, 0, +1\}$.

Last, the 2-operation: (2.2) and $(z_1 + z_2)^{(2)} = z_1^{(2)} + [z_1, z_2] + z_2^{(2)}$ (cf. Definition 2.7(e)) give

$$(\underline{\xi}^{\underline{a}} \partial_j)^{(2)} = 0, \quad (\underline{\xi}^{\underline{b}} (\xi_\ell \partial_\ell - \xi_t \partial_t))^{(2)} = 0, \quad \forall \underline{a}, \underline{b} : |\underline{a}| \in 2\mathbb{N}, |\underline{b}| \in (2\mathbb{N} + 1) \quad (2.10)$$

2.14. Definition of $\tilde{S}(n)$. We retain notations of Definitions 2.10 and 2.12 above. In addition, we assume now that n is even and $n \geq 4$. Define

$$\tilde{S}(n) := \left\{ D \in W(n) \mid (1 + \xi_1 \cdots \xi_n) \operatorname{div}(D) + D(\xi_1 \cdots \xi_n) = 0 \right\}$$

In order to describe $\tilde{S}(n)$, write $D \in W(n)$ as $D := \sum_{z=-1}^{n-1} D_z$ with $D_z \in W(n)_z$. The defining equation of $\tilde{S}(n)$ takes place in the graded superalgebra $\Lambda(n) = \bigoplus_{z=0}^n \Lambda(n)_z$: when we single out the different homogeneous summands the left-hand side of this equation reads

$$(1 + \xi_1 \cdots \xi_n) \operatorname{div}(D) + D(\xi_1 \cdots \xi_n) = \operatorname{div}(D_0) + \cdots + \operatorname{div}(D_{n-2}) + \\ + (\operatorname{div}(D_{n-1}) + D_{-1}(\xi_1 \cdots \xi_n)) + (D_0(\xi_1 \cdots \xi_n) + \xi_1 \cdots \xi_n \operatorname{div}(D_0))$$

(each $\operatorname{div}(D_z)$ is homogeneous of degree z , in particular $\operatorname{div}(D_{-1}) = 0$). Thus the defining equation $(1 + \xi_1 \cdots \xi_n) \operatorname{div}(D) + D(\xi_1 \cdots \xi_n) = 0$ of $\tilde{S}(n)$ is equivalent to the system

$$\circledast : \begin{cases} \operatorname{div}(D_z) = 0 & (\in \Lambda(n)_z) & \forall 0 \leq z \leq n-2 \\ \operatorname{div}(D_{n-1}) + D_{-1}(\xi_1 \cdots \xi_n) = 0 & (\in \Lambda(n)_{n-1}) \\ D_0(\xi_1 \cdots \xi_n) + \xi_1 \cdots \xi_n \operatorname{div}(D_0) = 0 & (\in \Lambda(n)_n) \end{cases}$$

The solution of the system \circledast is immediate: $D_z \in \operatorname{Ker}(\operatorname{div}) \cap W(n)_z = S(n)_z$, for $0 \leq z \leq n-2$, for the first $n-1$ equations, while the last one has solution $D_0 \in \operatorname{Ker}(\operatorname{div}) \cap W(n)_0 = S(n)_0$, hence it is redundant. For the last but one, let us write $D_{n-1} := \sum_{j=1}^n c_j \xi_1 \cdots \xi_n \partial_j$ and $D_{-1} := \sum_{j=1}^n d_j \partial_j$: then these yield solutions of that equation if and only if $c_j + d_j = 0$ for all $j = 1, \dots, n$, in other words if and only if $D_{-1} + D_{n-1} = \sum_{j=1}^n c_j (\xi_1 \cdots \xi_n - 1) \partial_j$. The outcome is that

$$\tilde{S}(n) = \bigoplus_{[z] \in \mathbb{Z}_n} \tilde{S}(n)_{[z]}, \quad \begin{aligned} \tilde{S}(n)_{[z]} &:= S(n)_z & \forall 0 \leq z \leq n-2 \\ \tilde{S}(n)_{[n-1]} &:= \operatorname{Span}_{\mathbb{K}}((\xi_1 \cdots \xi_n - 1) \partial_j)_{j=1, \dots, n} \end{aligned} \quad (2.11)$$

which is a splitting of $\tilde{S}(n)$ as a \mathbb{Z}_n -graded Lie superalgebra — see below for more details. Moreover, the natural \mathbb{Z} -filtration of $W(n)$ induces a similar filtration on $\tilde{S}(n)$: the associated graded Lie superalgebra then is graded-isomorphic to $S(n)$. Finally, the \mathbb{Z}_n -grading of $\tilde{S}(n)$ is consistent with the \mathbb{Z}_2 -grading, again in the obvious sense (like for $W(n)$ and $S(n)$).

Remark: to have a uniform notation we shall also write $\tilde{S}(n)_z := \tilde{S}(n)_{[z]}$ for all $-1 \leq z \leq n-2$ and $\tilde{S}(n)_z := \{0\}$ for all $z \in \mathbb{Z} \setminus \{-1, \dots, n-2\}$. Then $\tilde{S}(n) = \bigoplus_{z \in \mathbb{Z}} \tilde{S}(n)_z$ as a vector space.

The results we found for $W(n)$ — cf. Definition 2.10 — and $S(n)$ — cf. Definition 2.12 — give:

(a) a basis of the \mathbb{K} -vector space $\tilde{S}(n)$ is given by the union of the set $\bigcup_{z=0}^{n-2} B_{S(n); z}$ with the set $\{(\xi_1 \cdots \xi_n - 1) \partial_j\}_{j=1, \dots, n}$; in detail, it is

$$B_{\tilde{S}(n)} := \left\{ \underline{\xi}^{\underline{e}} \partial_i \mid \begin{smallmatrix} \underline{e}(i)=0 \\ |\underline{e}|>0 \end{smallmatrix} \right\} \cup \left\{ \underline{\xi}^{\underline{e}} (\xi_j \partial_j - \xi_{j'} \partial_{j'}) \mid \begin{smallmatrix} 1 \leq j < j' \leq n-1, \\ \underline{e}(j)=0=\underline{e}(j'), \\ \underline{e}(j'')=1 \quad \forall j < j'' < j' \end{smallmatrix} \right\} \cup \left\{ (\xi_1 \cdots \xi_n - 1) \partial_j \right\}_{1 \leq j \leq n}$$

Again, this basis is homogeneous (for both the \mathbb{Z}_n -grading and the \mathbb{Z}_2 -grading), i.e. $B_{\tilde{S}(n)} = \bigcup_{[z] \in \mathbb{Z}_n} B_{\tilde{S}(n); [z]}$ and $B_{\tilde{S}(n)} = \bigcup_{\bar{z} \in \mathbb{Z}_2} B_{\tilde{S}(n); \bar{z}}$ where $B_{\tilde{S}(n); [z]} := B_{\tilde{S}(n)} \cap \tilde{S}(n)_{[z]}$, respectively $B_{\tilde{S}(n); \bar{z}} := B_{\tilde{S}(n)} \cap \tilde{S}(n)_{\bar{z}}$, is a basis of $\tilde{S}(n)_{[z]}$, respectively of $\tilde{S}(n)_{\bar{z}}$, for every $[z] \in \mathbb{Z}_n$, $\bar{z} \in \mathbb{Z}_2$.

(b) $\tilde{S}(n)_{[0]}$ is a Lie subalgebra of the even part $\tilde{S}(n)_0$ of $\tilde{S}(n)$; as it coincides with $S(n)_0$, it is (again) isomorphic to $\mathfrak{sl}(n)$, see §2.12;

(c) $\tilde{S}(n)_{[-1]}$, as a module for $\tilde{S}(n)_0 \cong \mathfrak{sl}(n)$, is the dual of the standard module of $\mathfrak{sl}(n)$.

2.15. The Lie structure in $\tilde{S}(n)$. To describe the Lie (super)structure of $\tilde{S}(n)$ we can use explicit formulas for the Lie bracket and the 2-operation of elements in $B_{\tilde{S}(n)}$. Part of this basis is a subset of the basis $B_{S(n)}$ of $S(n)$ in §2.12, thus for these elements we refer to formulas therein.

We look now at the remaining cases. The first case is

$$\begin{aligned} \left[(\xi_1 \cdots \xi_n - 1) \partial_i, (\xi_1 \cdots \xi_n - 1) \partial_j \right] &= (-1)^j \xi_1 \cdots \hat{\xi}_j \cdots \xi_n \partial_i + (-1)^i \xi_1 \cdots \hat{\xi}_i \cdots \xi_n \partial_j = \\ &= \begin{cases} (-1)^{i+j-1} \xi_1 \cdots \hat{\xi}_i \cdots \hat{\xi}_j \cdots \xi_n (\xi_i \partial_i - \xi_j \partial_j) & \forall i < j \\ 2(-1)^j \xi_1 \cdots \hat{\xi}_j \cdots \xi_n \partial_j & \forall i = j \\ (-1)^{i+j-1} \xi_1 \cdots \hat{\xi}_j \cdots \hat{\xi}_i \cdots \xi_n (\xi_j \partial_j - \xi_i \partial_i) & \forall i > j \end{cases} \end{aligned}$$

The second case splits in turn into several subcases. Namely, the first two subcases are

$$\begin{aligned} \left[(\xi_1 \cdots \xi_n - 1) \partial_j, \xi^{\underline{e}} \partial_i \right] &= \begin{cases} \pm \xi^{\underline{e}-\underline{e}_j} \partial_j & \text{if } \underline{e}(j) = 1 \\ 0 & \text{if } \underline{e}(j) = 0 \end{cases} \quad \text{for } |\underline{e}| > 1 \text{ with } \underline{e}(i) = 0, \\ \left[(\xi_1 \cdots \xi_n - 1) \partial_j, \xi_k \partial_i \right] &= \delta_{j,k} (\xi_1 \cdots \xi_n - 1) \partial_i \quad \text{for } |\underline{e}| = 1 \text{ with } k \neq i. \end{aligned}$$

The third subcase is

$$\begin{aligned} \left[(\xi_1 \cdots \xi_n - 1) \partial_j, \xi^{\underline{e}} (\xi_h \partial_h - \xi_k \partial_k) \right] &= \\ &= \begin{cases} \pm \xi^{\underline{e}-\underline{e}_j} (\xi_h \partial_h - \xi_k \partial_k) + (-1)^{|\underline{e}|+1} (\delta_{j,h} - \delta_{j,k}) \xi^{\underline{e}} \partial_j & \text{if } \underline{e}(j) = 1 \\ (-1)^{|\underline{e}|+1} (\delta_{j,h} - \delta_{j,k}) \xi^{\underline{e}} \partial_j & \text{if } \underline{e}(j) = 0 \end{cases} \end{aligned}$$

where $\underline{e}(h) = 0 = \underline{e}(k)$, $|\underline{e}| > 0$ (with $k = h + 1$ if we want the second element to belong to $B_{S(n)}$ — yet the formula above holds in general for any h and k). The fourth, last subcase is

$$\left[(\xi_1 \cdots \xi_n - 1) \partial_j, (\xi_h \partial_h - \xi_k \partial_k) \right] = (\delta_{j,h} - \delta_{j,k}) (\xi_1 \cdots \xi_n - 1) \partial_j$$

Finally, for the 2-operation we have that (2.2) and the identity in Definition 2.7(e) give

$$((\xi_1 \cdots \xi_n - 1) \partial_j)^{(2)} = (-1)^j \xi_1 \cdots \hat{\xi}_j \cdots \xi_n \partial_j \quad \forall j = 1, \dots, n. \quad (2.12)$$

2.16. Definition of $H(n)$. We retain again notations of Definition 2.10 above, with $n \geq 4$.

Let $\underline{\omega} := (\omega_{i,j})_{i=1, \dots, n; j=1, \dots, n;}$ be a symmetric, non-singular square matrix of order n with entries in $\Lambda(n)$: this defines canonically a symplectic form in $\Lambda(n)$ which we still denote by $\underline{\omega}$, namely $\underline{\omega} := \sum_{i,j=1}^n \omega_{i,j} d\xi_i \circ d\xi_j$ — cf. [14], §3.3. For any such form and any $D := \sum_{i=1}^n P_i(\underline{\xi}) \partial_i \in W(n)$ the form $D\underline{\omega}$ is naturally defined, and we set

$$\tilde{H}(\underline{\omega}) := \left\{ D \in W(n) \mid D\underline{\omega} = 0 \right\}$$

This is a Lie subsuperalgebra of $W(n)$. We define a special Lie subsuperalgebra of $\tilde{H}(\underline{\omega})$, namely

$$H(\underline{\omega}) := [\tilde{H}(\underline{\omega}), \tilde{H}(\underline{\omega})]$$

All Lie superalgebras $\tilde{H}(\underline{\omega})$, for different forms $\underline{\omega}$, are isomorphic with each other; the same holds for the various $H(\underline{\omega})$. Thus we can fix a specific form of the matrix $\underline{\omega}$: we choose it to be

$$\underline{\omega} := \begin{pmatrix} 0_r & I_r \\ I_r & 0_r \end{pmatrix} \quad \text{if } n = 2r, \quad \underline{\omega} := \begin{pmatrix} 0_{r \times r} & I_{r \times r} & 0_{r \times 1} \\ I_{r \times r} & 0_{r \times r} & 0_{r \times 1} \\ 0_{1 \times r} & 0_{1 \times r} & 1 \end{pmatrix} \quad \text{if } n = 2r + 1 \quad (2.13)$$

(where $I_{r \times r}$ is the identity matrix of order r , and so on), so that the corresponding form is

$$\underline{\omega} = \sum_{i=1}^r (d\xi_i \circ d\xi_{r+i} + d\xi_{r+i} \circ d\xi_i) \quad \text{if } n = 2r, \quad \underline{\omega} = \sum_{i=1}^r (d\xi_i \circ d\xi_{r+i} + d\xi_{r+i} \circ d\xi_i) + d\xi_n \circ d\xi_n \quad \text{if } n = 2r + 1.$$

For this specific choice of form $\underline{\omega}$, we use hereafter the notation $\tilde{H}(n) := \tilde{H}(\underline{\omega})$, $H(n) := H(\underline{\omega})$.

The natural \mathbb{Z} -filtration on $W(n)$ induces a \mathbb{Z} -filtration on $\tilde{H}(\underline{\omega})$ and $H(\underline{\omega})$, for any $\underline{\omega}$. Even more, on $\tilde{H}(n)$ and $H(n)$ the \mathbb{Z} -grading on $W(n)$ induces \mathbb{Z} -gradings as well. Then the graded Lie superalgebra associated with $\tilde{H}(\underline{\omega})$, for any $\underline{\omega}$, is isomorphic to $H(n)$ as a graded Lie superalgebra. Several properties of these graded Lie superalgebras are recorded in [14], §3. Here we just recall

$$\tilde{H}(n) = \bigoplus_{z \in \mathbb{Z}} \tilde{H}(n)_z, \quad \tilde{H}(n)_z = W(n)_z \cap \tilde{H}(n), \quad H(n) = \bigoplus_{z \in \mathbb{Z}} H(n)_z, \quad H(n)_z = W(n)_z \cap H(n)$$
 where $\tilde{H}(n)_z \neq \{0\}$ iff $-1 \leq z \leq n-2$, $H(n)_z \neq \{0\}$ iff $-1 \leq z \leq n-3$. Moreover $\tilde{H}(n)_z = H(n)_z$ for $-1 \leq z \leq n-3$ and $\dim(\tilde{H}(n)_{n-2}) = 1$: in particular, $\tilde{H}(n) = H(n) \oplus \tilde{H}(n)_{n-2}$.

Like for $W(n)$ and $S(n)$, this \mathbb{Z} -grading yields also gradings by cyclic groups, namely $\tilde{H}(n) = \bigoplus_{[z] \in \mathbb{Z}_n} \tilde{H}(n)_{[z]}$ and $H(n) = \bigoplus_{[z] \in \mathbb{Z}_{n-1}} H(n)_{[z]}$ with $\tilde{H}(n)_{[z]} := \bigoplus_{\zeta \equiv z \pmod n} \tilde{H}(n)_\zeta$ for all $[z] \in \mathbb{Z}_n$ and $H(n)_{[z]} := \bigoplus_{\zeta \equiv z \pmod{n-1}} H(n)_\zeta$ for all $[z] \in \mathbb{Z}_{n-1}$ (the integers modulo $n-1$). Again, in both cases the \mathbb{Z} -grading yields a \mathbb{Z} -filtration, coinciding with the one induced by $W(n)$, whose associated graded Lie superalgebra is isomorphic to $H(n)$ and $\tilde{H}(n)$ respectively. Finally (as for W , \tilde{S} and S) the \mathbb{Z} -grading is consistent with the \mathbb{Z}_2 -grading both in $\tilde{H}(n)$ and in $H(n)$, in the obvious sense.

To describe $\tilde{H}(n)$, $H(n)$ and their graded summands, we exploit a different realization of them.

For any given closed differential form $\underline{\omega}$ as above, consider on $V_n := \text{Span}(\xi_1, \dots, \xi_n)$ the bilinear form corresponding to $\underline{\omega}$, and take on the vector space $\Lambda(n)$ the structure of Clifford algebra associated to V_n with such a form. Then the supercommutator on $\Lambda(n)$ reads

$$\{f, g\} := (-1)^{p(f)} \sum_{i,j=1}^n \dot{\omega}_{i,j} \partial_i(f) \partial_j(g) \quad (2.14)$$

(we use braces instead of square brackets for psychological reasons) where $\underline{\dot{\omega}} = (\dot{\omega}_{i,j})_{i,j=1,\dots,n}^{j=1,\dots,n,i=1,\dots,n} = \underline{\omega}^{-1}$ is the inverse of the matrix $\underline{\omega}$. If we consider on $\Lambda(n)$ its natural associative product and the Lie superbracket in (2.14), it is a (supercommutative) Poisson superalgebra, which we denote by $P_{\underline{\omega}}(n)$. By the analog to Poincaré's lemma, there exists a Lie superalgebra epimorphism

$$\phi : P_{\underline{\omega}}(n) \longrightarrow \tilde{H}(\underline{\omega}), \quad f \mapsto D_f := \sum_{i,j=1}^n \dot{\omega}_{i,j} \partial_i(f) \partial_j \quad (2.15)$$

which shifts the \mathbb{Z} -grading by -2 , i.e. $\phi(P_{\underline{\omega}}(n)_z) = \tilde{H}(\underline{\omega})_{z-2}$ for all z (so the induced \mathbb{Z}_2 -grading is preserved) and has kernel the \mathbb{K} -span of 1 ; so $P_{\underline{\omega}}(n)/\mathbb{K} \cdot 1_{P_{\underline{\omega}}(n)} \cong \tilde{H}(\underline{\omega})$ via an isomorphism induced by ϕ . Moreover, the restriction of ϕ to $\{f \in P_{\underline{\omega}}(n) = \Lambda(n) \mid \epsilon(f) = 0\}$, where $\epsilon(f)$ is the constant term in f (thought of as a skew-polynomial in the ξ_i 's), is a bijection: thus we have

$$\tilde{H}(\underline{\omega}) = \{D_f \mid \epsilon(f) = 0\} \quad \text{and} \quad [D_f, D_g] = D_{\{f,g\}} \quad (2.16)$$

The outcome is that we can describe $\tilde{H}(\underline{\omega})$ via the isomorphism of it with $P_{\underline{\omega}}(n)/\mathbb{K} \cdot 1_{P_{\underline{\omega}}(n)}$, for which we can compute the Lie superbracket using (2.14). We do it now for the canonical $\underline{\omega}$.

Let $\underline{\omega}$ be the canonical matrix chosen as in (2.13). Then we write $P(n) := P_{\underline{\omega}}(n)$ for the corresponding Poisson superalgebra. In this case (2.14) and (2.15) take the simpler form

$$\{f, g\} := (-1)^{p(f)} \left(\sum_{s=1}^r (\partial_s(f) \partial_{r+s}(g) + \partial_{r+s}(f) \partial_s(g)) + \delta_{n \in (2\mathbb{N}+1)} \partial_{2r+1}(f) \partial_{2r+1}(g) \right) \quad (2.17)$$

$$f \mapsto D_f := \sum_{s=1}^r (\partial_s(f) \partial_{r+s} + \partial_{r+s}(f) \partial_s) + \delta_{n \in (2\mathbb{N}+1)} \partial_{2r+1}(f) \partial_{2r+1} \quad (2.18)$$

where $\delta_{n \in (2\mathbb{N}+1)} := 1$ if n is odd (written as $n = 2r + 1$) and $\delta_{n \in (2\mathbb{N}+1)} := 0$ otherwise.

For each $z \in \mathbb{Z}$ the set $B_{P(n);z} := \{\xi^{\underline{e}} \mid \underline{e} \in \{0,1\}^n, |\underline{e}| = z\}$ is a \mathbb{K} -basis of $P(n)_z$, and for each $\bar{z} \in \mathbb{Z}_2$ the set $B_{P(n);\bar{z}} := \bigcup_{(z \bmod 2) = \bar{z}} B_{P(n);z}$ is a \mathbb{K} -basis of $P(n)_{\bar{z}}$. It follows that $B_{P(n)} := \bigcup_{z \in \mathbb{Z}} B_{P(n);z}$ is a \mathbb{K} -basis (\mathbb{Z} -homogeneous and \mathbb{Z}_2 -homogeneous) of $P(n)$. Applying ϕ ,

we get bases for $\tilde{H}(n)$, $H(n)$ and their graded summands. Focusing on $H(n)$, we find:

(a) a basis of the \mathbb{K} -vector space $H(n)$ is given by

$$B_{H(n)} := \left\{ D_{\xi^{\underline{e}}} \mid \underline{e} \in \{0,1\}^n, 0 < |\underline{e}| < n \right\}$$

This basis is homogeneous (for the \mathbb{Z} -grading and the \mathbb{Z}_2 -grading), i.e. $B_{H(n)} = \bigcup_{z \in \mathbb{Z}} B_{H(n);z}$ and $B_{H(n)} = \bigcup_{\bar{z} \in \mathbb{Z}_2} B_{H(n);\bar{z}}$ where $B_{H(n);z} := B_{H(n)} \cap H(n)_z$ is a basis of $H(n)_z$ and $B_{H(n);\bar{z}} := B_{H(n)} \cap H(n)_{\bar{z}}$ is a basis of $H(n)_{\bar{z}}$, for every $z \in \mathbb{Z}$ and $\bar{z} \in \mathbb{Z}_2$.

(b) $H(n)_0$ is a Lie subalgebra of the even part $H(n)_{\bar{0}}$ of $H(n)$, isomorphic to $\mathfrak{so}(n)$, the latter being considered with respect to the canonical form $\underline{\omega}$; an isomorphism is given by

$$\begin{aligned} D_{\xi_h \xi_k} &= \xi_k \partial_{r+h} - \xi_h \partial_{r+k} \mapsto e_{k,r+h} - e_{h,r+k} & \forall \quad 1 \leq h < k \leq r \\ D_{\xi_h \xi_{r+k}} &= \xi_{r+k} \partial_{r+h} - \xi_h \partial_k \mapsto e_{r+k,r+h} - e_{h,k} & \forall \quad 1 \leq h \leq r, \quad 1 \leq k \leq r \\ D_{\xi_{r+h} \xi_{r+k}} &= \xi_{r+k} \partial_h - \xi_{r+h} \partial_k \mapsto e_{r+k,h} - e_{r+h,k} & \forall \quad 1 \leq h < k \leq r \\ D_{\xi_t \xi_{2r+1}} &= \xi_{2r+1} \partial_{r+t} - \xi_t \partial_{2r+1} \mapsto e_{2r+1,r+t} - e_{t,2r+1} & \forall \quad 1 \leq t \leq r, \quad n = 2r+1 \\ D_{\xi_{r+t} \xi_{2r+1}} &= \xi_{2r+1} \partial_t - \xi_{r+t} \partial_{2r+1} \mapsto e_{2r+1,r+t} - e_{t,2r+1} & \forall \quad 1 \leq t \leq r, \quad n = 2r+1 \end{aligned}$$

(with notation as in Definition 2.10), the last two formulas being in use only for odd $n = 2r+1$.

(c) $H(n)_{-1}$, as a module for $H(n)_0 \cong \mathfrak{so}(n)$, is the dual of the standard module of $\mathfrak{so}(n)$.

2.17. The Lie structure in $H(n)$. We describe now the Lie (super)structure of $H(n)$ in terms of its basis $B_{H(n)}$. We make use of the isomorphism $P(n)/\mathbb{K} \cdot 1_{P(n)} \cong \tilde{H}(n)$ along with formulas (2.16–17) and the fact that $\tilde{H}(n) = H(n) \oplus \tilde{H}(n)_{n-2}$ with $\tilde{H}(n)_{n-2} = \mathbb{K} \cdot D_{\xi_1 \xi_2 \dots \xi_n}$. In short, we have to compute the brackets $\{\xi^{\underline{a}}, \xi^{\underline{b}}\}$ in $P(n)$ for all $\underline{a}, \underline{b} \in \{0, 1\}^n$ such that $0 < |\underline{a}|, |\underline{b}| < n$.

By (2.17) we have

$$\{\xi^{\underline{a}}, \xi^{\underline{b}}\} = (-1)^{|\underline{a}|} \left(\sum_{s=1}^r \left(\partial_s(\xi^{\underline{a}}) \partial_{r+s}(\xi^{\underline{b}}) + \partial_{r+s}(\xi^{\underline{a}}) \partial_s(\xi^{\underline{b}}) \right) + \delta_{n \in (2\mathbb{N}+1)} \partial_{2r+1}(\xi^{\underline{a}}) \partial_{2r+1}(\xi^{\underline{b}}) \right)$$

With a detailed (yet elementary) analysis, one finds only two possibilities. The first one is

$$\exists s : \underline{a}(s) = 1 = \underline{b}(r+s), \underline{a}(r+s) = 1 = \underline{b}(s) \implies \{\xi^{\underline{a}}, \xi^{\underline{b}}\} = 0 \quad (2.19)$$

On the other hand, the second possibility is either

$$\# s : \left. \begin{aligned} &\underline{a}(s) = 1 = \underline{b}(r+s), \underline{a}(r+s) = 1 = \underline{b}(s) \\ &(\underline{a}(2r+1), \underline{b}(2r+1)) \neq (1, 1) \end{aligned} \right\} \implies \{\xi^{\underline{a}}, \xi^{\underline{b}}\} = \sum_{k=1}^r \eta_k \xi^{\underline{a}+\underline{b}-\underline{e}_k-\underline{e}_{r+k}} \quad (2.20)$$

for some $\eta_k \in \{+1, 0, -1\}$, or (only possible if n is odd, written as $n = 2r+1$)

$$\# s : \left. \begin{aligned} &\underline{a}(s) = 1 = \underline{b}(r+s), \underline{a}(r+s) = 1 = \underline{b}(s) \\ &(\underline{a}(2r+1), \underline{b}(2r+1)) = (1, 1) \end{aligned} \right\} \implies \{\xi^{\underline{a}}, \xi^{\underline{b}}\} = \eta_{2r+1} \xi^{\underline{a}+\underline{b}-2\underline{e}_{2r+1}} \quad (2.21)$$

for some $\eta_{2r+1} \in \{+1, 0, -1\}$. For later use, we also record the following fact

$$\left. \begin{aligned} &\underline{a}(s) = \underline{a}(r+s), \underline{b}(s) = \underline{b}(r+s) \quad \forall s \\ &\underline{a}(2r+1) = 0 = \underline{b}(2r+1) \end{aligned} \right\} \implies \{\xi^{\underline{a}}, \xi^{\underline{b}}\} = 0 \quad (2.22)$$

also proved by straightforward inspection. Similar results, still proved by direct analysis, are

$$\{\xi^{\underline{a}}, \{\xi^{\underline{a}}, \xi^{\underline{b}}\}\} = 0 \quad \forall \underline{a}, \underline{b} \in \{0, 1\}^n : |\underline{a}| > 3 \quad (2.23)$$

$$\{\xi_{2r+1}, \xi_{2r+1}\} = -1, \quad \{\xi^{\underline{a}}, \xi^{\underline{a}}\} = 0 \quad \forall \underline{a} \in \{0, 1\}^n \setminus \{\underline{e}_{2r+1}\} \quad (2.24)$$

All the formulas above yield also the Lie brackets among elements of $B_{H(n)}$, via the identity $[D_{\xi^{\underline{a}}}, D_{\xi^{\underline{b}}}] = D_{[\xi^{\underline{a}}, \xi^{\underline{b}}]}$ — see formula (2.16). Similarly, from these formulas and from the identity in Definition 2.7(e), taking also (2.24) into account, we get for the 2-operation the formulas

$$D_{\xi^{\underline{a}}}^{(2)} = 0 \quad \forall \underline{a} \in \{0, 1\}^n \quad (2.25)$$

\diamond_{\perp} From now on, \mathfrak{g} will be a Lie superalgebra of Cartan type: $W(n)$, $S(n)$, $\tilde{S}(n)$ or $H(n)$. \diamond_{\perp}

2.18. Cartan subalgebras, roots, root spaces. Let \mathfrak{g} be a Lie superalgebra of Cartan type. We call *Cartan subalgebras of \mathfrak{g}* the Cartan subalgebras of the reductive Lie algebra \mathfrak{g}_0 , which is $\mathfrak{gl}(n)$, $\mathfrak{sl}(n)$ or $\mathfrak{so}(n)$ respectively if \mathfrak{g} is $W(n)$, $S(n)$ or $\tilde{S}(n)$, or $H(n)$. We fix one of them (the “standard” one), namely $\mathfrak{h} := \text{Span}_{\mathbb{K}}(\{\xi_k \partial_k\}_{1 \leq k \leq n})$ for case $W(n)$, $\mathfrak{h} := \text{Span}_{\mathbb{K}}(\{(\xi_k \partial_k - \xi_{k+1} \partial_{k+1})\}_{1 \leq k \leq n-1})$ for cases $S(n)$ and $\tilde{S}(n)$, and $\mathfrak{h} := \text{Span}_{\mathbb{K}}(\{(\xi_k \partial_k - \xi_{r+k} \partial_{r+k})\}_{1 \leq k \leq r})$ for case $H(n)$, with $r := \lfloor n/2 \rfloor$. In all cases, the spanning set we considered in \mathfrak{h} is actually a \mathbb{K} -basis.

Now consider the element $\mathcal{E} := \sum_{i=1}^n \xi_i \partial_i \in W(n)_0$: we set $\bar{\mathfrak{h}} := \mathfrak{h}$, $\bar{\mathfrak{g}}_0 := \mathfrak{g}_0$, $\bar{\mathfrak{g}} := \mathfrak{g}$ when \mathfrak{g} is of type W or \tilde{S} , and $\bar{\mathfrak{h}} := \mathfrak{h} + \mathbb{K}\mathcal{E}$, $\bar{\mathfrak{g}}_0 := \mathfrak{g}_0 + \mathbb{K}\mathcal{E}$, $\bar{\mathfrak{g}} := \mathfrak{g} + \mathbb{K}\mathcal{E}$ when \mathfrak{g} is of type S or H . In this way $\mathcal{E} \in \bar{\mathfrak{g}}$ iff \mathfrak{g} is \mathbb{Z} -graded, with $[\mathcal{E}, X] = zX$ for $X \in \mathfrak{g}_z$ (cf. [14], §1.2.12 and §4.1.2).

The Cartan subalgebra $\bar{\mathfrak{h}}$ acts by adjoint action on $\bar{\mathfrak{g}}$, and \mathfrak{g} itself is an $\bar{\mathfrak{h}}$ -submodule. Thus we have a decomposition of \mathfrak{g} into weight spaces for the $\bar{\mathfrak{h}}$ -action, namely (cf. [16], §4)

$$\mathfrak{g} = \bigoplus_{\alpha \in \bar{\mathfrak{h}}^*} \mathfrak{g}_\alpha, \quad \mathfrak{g}_\alpha := \{w \in \mathfrak{g} \mid [h, w] = \alpha(h)w, \forall h \in \bar{\mathfrak{h}}\}$$

called *root (space) decomposition* of \mathfrak{g} , where $\mathfrak{g}_{\alpha=0} = \mathfrak{h}$. The terminology for such a context is standard: $\Delta := \{\alpha \in \bar{\mathfrak{h}}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$ is the *root system* of \mathfrak{g} , its elements are called *roots*, we have *root spaces* (each \mathfrak{g}_α for $\alpha \neq 0$), *root vectors*, etc. For every root α one has either $\mathfrak{g}_\alpha \subseteq \bar{\mathfrak{g}}_0$ or $\mathfrak{g}_\alpha \subseteq \bar{\mathfrak{g}}_1$: accordingly, we call α *even* or *odd*; we set $\Delta_{\bar{z}} := \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subseteq \bar{\mathfrak{g}}_{\bar{z}}\}$ for $\bar{z} \in \{\bar{0}, \bar{1}\}$.

We call a root α *essential* if $-\alpha \in \Delta$, and *nonessential* if $-\alpha \notin \Delta$.

Finally, the *multiplicity* of a root α , by definition, is the non-negative integer $\mu(\alpha) := \dim(\mathfrak{g}_\alpha)$; then we call the root α respectively *thin* or *thick* if $\mu(\alpha) = 1$ or $\mu(\alpha) > 1$.

We denote by Q the \mathbb{Z} -lattice $Q := \mathbb{Z}\Delta$ spanned by Δ inside $\bar{\mathfrak{h}}^*$. The root space decomposition is a Q -grading of \mathfrak{g} (as a Lie superalgebra). This Q -grading is compatible with the \mathbb{Z} -grading $\mathfrak{g} = \bigoplus_{z \in \mathbb{Z}} \mathfrak{g}_z$ (only as vector space for \mathfrak{g} of type \tilde{S} : cf. the *Remark* in §2.14), in the following sense: one has $\mathfrak{h} \subseteq \mathfrak{g}_0$ and for each root α also $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{ht(\alpha)}$ for a unique integer $ht(\alpha) \in \mathbb{Z}$, so that

$$\mathfrak{g}_0 = \bigoplus_{\alpha: ht(\alpha)=0} \mathfrak{g}_\alpha \oplus \mathfrak{h}, \quad \mathfrak{g}_z = \bigoplus_{\alpha: ht(\alpha)=z} \mathfrak{g}_\alpha \quad (\forall z \in \mathbb{Z} \setminus \{0\})$$

The unique integer $ht(\alpha) \in \mathbb{Z}$ thus associated with every root α is called the *height* of α . As another consequence, we can also partition the set of roots according to the height, namely, $\Delta = \coprod_{z \in \mathbb{Z}} \Delta_z$ with $\Delta_z := \{\alpha \in \Delta \mid ht(\alpha) = z\}$; then $\Delta_{\bar{0}} = \coprod_{z \in 2\mathbb{Z}} \Delta_z$ and $\Delta_{\bar{1}} = \coprod_{z \in (2\mathbb{Z}+1)} \Delta_z$.

We set $\Delta_{\bar{0}\uparrow} := \Delta_{\bar{0}} \setminus \Delta_0$, $\Delta_{\bar{1}\uparrow} := \Delta_{\bar{1}} \setminus \Delta_{-1}$ and $\tilde{\Delta} := \{(\alpha, j)\}_{1 \leq j \leq \mu(\alpha)}^{\alpha \in \Delta}$; we denote $\pi: \tilde{\Delta} \longrightarrow \Delta$ the map $\tilde{\alpha} = (\alpha, k) \mapsto \alpha$, and $\tilde{\Delta}_{\hat{z}} := \pi^{-1}(\Delta_{\hat{z}})$ for $\hat{z} \in \{\bar{0}, \bar{1}, \bar{0}^\uparrow, \bar{1}^\uparrow\} \cup \{-1, 0, 1, \dots, n\}$.

We conclude introducing (or recalling) the notion of *coroot* adfsassociated with a classical root.

When \mathfrak{g} is of type W , S or H , the Lie algebra $\bar{\mathfrak{g}}_0$ is reductive, of the form $\bar{\mathfrak{g}}_0 = \bar{\mathfrak{g}}_0^{ss} \oplus \mathbb{K}\mathcal{E}$ where $\bar{\mathfrak{g}}_0^{ss}$ is its semisimple part (actually simple) and $\mathbb{K}\mathcal{E}$ is its radical (actually the centre); similarly $\bar{\mathfrak{h}}$ splits as $\bar{\mathfrak{h}} = \bar{\mathfrak{h}}_{ss} \oplus \mathbb{K}\mathcal{E}$ with $\bar{\mathfrak{h}}_{ss} := \bar{\mathfrak{h}} \cap \bar{\mathfrak{g}}_0^{ss}$: explicitly, $\bar{\mathfrak{h}}_{ss}$ is the (standard) Cartan subalgebra of $\bar{\mathfrak{g}}_0^{ss} \cong \mathfrak{sl}_n$ — if \mathfrak{g} is of type $W(n)$ or $S(n)$ — or of $\mathfrak{g}_0 \cong \mathfrak{so}_n$ — if \mathfrak{g} is of type $H(n)$.

When \mathfrak{g} is of type $\tilde{S}(n)$ the situation is simpler: the Lie algebra $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0$ is (semi)simple, isomorphic to \mathfrak{sl}_n . In order to unify notation, we write then $\bar{\mathfrak{g}}_0^{ss} := \bar{\mathfrak{g}}_0 = \mathfrak{g}_0$ and $\bar{\mathfrak{h}}_{ss} := \bar{\mathfrak{h}} = \mathfrak{h}$: explicitly, $\bar{\mathfrak{h}}_{ss} = \mathfrak{h}$ is the (standard) Cartan subalgebra of $\bar{\mathfrak{g}}_0 = \mathfrak{g}_0 \cong \mathfrak{sl}_n$.

As in any semisimple Lie algebra, the Killing form induce a \mathbb{K} -linear isomorphism $(\bar{\mathfrak{h}}_{ss})^* \xrightarrow{\cong} \bar{\mathfrak{h}}_{ss}$, denoted by $\gamma \mapsto t_\gamma$. When \mathfrak{g} is of type $\tilde{S}(n)$ we use this isomorphism to define (as usual: cf. [13]) the *coroot* $H_\alpha \in \bar{\mathfrak{h}}_{ss} = \bar{\mathfrak{h}}$ associated with any *classical* root $\alpha \in \Delta_0 (\subset (\bar{\mathfrak{h}}_{ss})^* = \bar{\mathfrak{h}}^*)$.

When \mathfrak{g} is of type W , S or H instead, note that every linear functional ϕ on $\bar{\mathfrak{h}}_{ss}$ uniquely extends to a linear functional on $\bar{\mathfrak{h}}$ such that $\phi(\mathcal{E}) = 0$: this yields an embedding of $(\bar{\mathfrak{h}}_{ss})^*$ into $\bar{\mathfrak{h}}^*$, so every root (in classical sense) of the simple Lie algebra $\bar{\mathfrak{g}}_0^{ss}$ identifies with an element of $\bar{\mathfrak{h}}^*$. Conversely, let $\delta \in \bar{\mathfrak{h}}^*$ be the unique \mathbb{K} -linear functional on $\bar{\mathfrak{h}} = \bar{\mathfrak{h}}_{ss} \oplus \mathbb{K}\mathcal{E}$ such that $\delta(\mathcal{E}) = 1$ and $\delta(\bar{\mathfrak{h}}_{ss}) = \{0\}$: coupled with the embedding $(\bar{\mathfrak{h}}_{ss})^* \hookrightarrow \bar{\mathfrak{h}}^*$, this yields $\bar{\mathfrak{h}}^* = (\bar{\mathfrak{h}}_{ss})^* \oplus \mathbb{K}\delta$. Now, using this last description of $\bar{\mathfrak{h}}^*$ and the isomorphism $(\bar{\mathfrak{h}}_{ss})^* \xrightarrow{\cong} \bar{\mathfrak{h}}_{ss}$ induced by the Killing form, we extend the latter to an isomorphism $\bar{\mathfrak{h}}^* \hookrightarrow \bar{\mathfrak{h}}$, ($\gamma \mapsto t_\gamma$, $\mathcal{E} \mapsto \delta$): via this, we define again the *coroot* $H_\alpha \in \bar{\mathfrak{h}}_{ss} \subsetneq \bar{\mathfrak{h}}$ associated with every *classical* root $\alpha \in \Delta_0 (\subset (\bar{\mathfrak{h}}_{ss})^* \subsetneq \bar{\mathfrak{h}}^*)$.

We describe now in detail all these objects; we distinguish cases W , S , \tilde{S} and H (cf. [16]).

Case $\mathfrak{g} = W(n)$: Let $\mathfrak{g} := W(n)$. Then $\mathfrak{g}_0 \cong \mathfrak{gl}_n$, and we fix the Cartan subalgebra $\mathfrak{h} = \bar{\mathfrak{h}}$ as above. Let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ be the standard basis of \mathfrak{h}^* : then the root system of \mathfrak{g} is given by

$$\Delta = \{ \varepsilon_{i_1} + \dots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n, j = 1, \dots, n \} \setminus \{0\}$$

From this description, one sees at once that every root $\alpha \in \Delta$ can be written as \pm the sum of “simple roots” chosen in a “simple root system”, just like for simple Lie superalgebras of *basic* type. For instance, one can take as “simple root system” the set $\Pi := \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$, which is even “distinguished” (i.e., it contains only one odd root, in this case ε_n).

If $\alpha = \sum_{s=1}^k \varepsilon_{i_s} - \varepsilon_j$ with $j \notin \{i_1, \dots, i_k\}$, then $\mathfrak{g}_{\alpha = \sum_{s=1}^k \varepsilon_{i_s} - \varepsilon_j}$ has \mathbb{K} -basis the singleton $\{\xi_{i_1} \dots \xi_{i_k} \partial_j\}$, so that $\mu(\alpha) = 1$; if instead $\alpha = \sum_{r=1}^h \varepsilon_{i_r}$, then $\mathfrak{g}_{\alpha = \sum_{r=1}^h \varepsilon_{i_r}}$ has \mathbb{K} -basis $\{\xi_{i_1} \dots \xi_{i_h} \xi_\ell \partial_\ell \mid \ell \in \{1, \dots, n\} \setminus \{i_1, \dots, i_h\}\}$, so $\mu(\alpha) = n - h$. Thus the *thick* roots are those of $\{\sum_{s=1}^h \varepsilon_{i_s} \mid h < n-1\}$ and the *thin* ones those of $\{\sum_{s=1}^k \varepsilon_{i_s} - \varepsilon_j \mid j \notin \{i_1, \dots, i_k\}\} \cup \{\sum_{s=1}^{n-1} \varepsilon_{i_s}\}$.

Now for any $\alpha = \varepsilon_{i_1} + \dots + \varepsilon_{i_k} - \varepsilon_j \in \Delta$, its *height* is easy to read off: it is given by $ht(\alpha) = k-1$.

If a root α is *even essential*, then $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \subseteq \mathfrak{g}_0$, the multiplicity of both α and $-\alpha$ is 1 and $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ together generate a Lie subalgebra isomorphic to $\mathfrak{sl}(2)$ inside \mathfrak{g}_0 . The set of even essential roots is $\{\varepsilon_i - \varepsilon_j\}_{i,j=1, \dots, n; i \neq j}$, the corresponding root spaces being $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \text{Span}_{\mathbb{K}}(\{\xi_i \partial_j\}) = \mathbb{K} \xi_i \partial_j$.

If instead α is *odd essential*, then either $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_{-1}$, $\mu(\alpha) = 1$, $\mu(-\alpha) = n-1$, or $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{g}_{-1}$, $\mu(-\alpha) = 1$, $\mu(\alpha) = n-1$. In the first case, $\alpha \in \{-\varepsilon_j\}_{1 \leq j \leq n}$ is thin while $-\alpha \in \{\varepsilon_j\}_{1 \leq j \leq n}$ is thick, and for each j the root spaces $\mathfrak{g}_{\alpha = -\varepsilon_j}$ and $\mathfrak{g}_{-\alpha = \varepsilon_j}$ have \mathbb{K} -basis $\{\partial_j\}$ and $\{\xi_j \xi_\ell \partial_\ell \mid 1 \leq \ell \leq n, \ell \neq j\}$ respectively (see above). In the second case the converse holds.

Finally the *nonessential* roots α are all those such that $ht(\alpha) > 1$ together with all those of the form $\alpha = \varepsilon_{i_1} + \varepsilon_{i_2} - \varepsilon_j$ (hence $ht(\alpha) = 1$) with $j \neq i_1, i_2$. \diamond

Case $\mathfrak{g} = S(n)$: Let $\mathfrak{g} := S(n)$. Now $\mathfrak{g}_0 \cong \mathfrak{sl}_n$, we fix the Cartan subalgebra \mathfrak{h} and set $\bar{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{K} \mathcal{E}$ as above. Clearly the root system of $\mathfrak{g} := S(n)$ is a subset of that of $W(n)$, namely

$$\Delta = \{ \varepsilon_{i_1} + \dots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n, k < n, j = 1, \dots, n \} \setminus \{0\}$$

in short, the roots of $S(n)$ are those of $W(n)$ whose height is less than $n-1$. In particular, the characterization of the height of a root of $S(n)$ is exactly the same as for $W(n)$.

By construction, root spaces of $S(n)$ or $W(n)$ enjoy the relation $S(n)_\alpha = W(n)_\alpha \cap S(n)$. The explicit description given for case W implies that $S(n)_\alpha = W(n)_\alpha$ when α is thin for $W(n)$, so that it is *thin* for $S(n)$ as well (the multiplicity being 1 in both cases). Instead, if a root α is of the form $\alpha = \sum_{s=1}^h \varepsilon_{i_s}$, with $h < n-1$ (so it is thick for $W(n)$), then the space $S(n)_\alpha$ has \mathbb{K} -basis $\{\xi^{\underline{i}} (\xi_j \partial_j - \xi_{j+1} \partial_{j+1}) \mid 1 \leq j \leq n-1, \underline{i}(j) = 0 = \underline{i}(j+1)\}$ with $\xi^{\underline{i}} := \xi_{i_1} \dots \xi_{i_h}$; so α in $S(n)$ has multiplicity $\mu(\alpha) = n-h-1$, hence α is *thick* for $S(n)$ if $h < n-2$, and it is *thin* if $h = n-2$.

Finally, it is clear that roots of $S(n)$ have a certain degree (for the \mathbb{Z}_2 -grading or the \mathbb{Z} -grading), and they are essential or non-essential, exactly as they have or they are for $W(n)$. \diamond

Case $\mathfrak{g} = \tilde{S}(n)$: Let $\mathfrak{g} := \tilde{S}(n)$. Like for $S(n)$, we have $\mathfrak{g}_0 \cong \mathfrak{sl}_n$ and we fix $\bar{\mathfrak{h}} := \mathfrak{h}$ with \mathfrak{h} as above. By the analysis in §§2.14, 2.15, we see that the root system of $\tilde{S}(n)$ is

$$\Delta = \{ \varepsilon_{i_1} + \dots + \varepsilon_{i_k} - \varepsilon_j \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n, k < n, j = 1, \dots, n \} \setminus \{0\}$$

like for $S(n)$, but now $\varepsilon_1 + \dots + \varepsilon_n = 0$ in $\bar{\mathfrak{h}}^* = \mathfrak{h}^*$. Also, for root spaces we have the following.

For every root α of $S(n)$ having non-negative height we have $\tilde{S}(n)_\alpha = S(n)_\alpha$. Instead, for the roots of height -1 (which are $-\varepsilon_1, \dots, -\varepsilon_n$) we have $\tilde{S}(n)_{-\varepsilon_j} = \text{Span}_{\mathbb{K}}((\xi_1 \dots \xi_n - 1) \partial_j)$ for all $j = 1, \dots, n$, with $\{(\xi_1 \dots \xi_n - 1) \partial_j\}$ being a \mathbb{K} -basis of any such root space.

It is worth stressing that the roots of the form $-\varepsilon_j$ are the only $\alpha \in \Delta$ such that $2\alpha \in \Delta$. Indeed, $2(-\varepsilon_j) = (\varepsilon_1 + \dots + \widehat{\varepsilon_j} + \dots + \varepsilon_n) - \varepsilon_j \in \Delta$ (for all $j = 1, \dots, n$), using the identity $\varepsilon_1 + \dots + \varepsilon_n = 0$ (in \mathfrak{h}^*); instead, direct analysis shows that $2\alpha \notin \Delta$ for all $\alpha \notin \{-\varepsilon_j\}_{j=1, \dots, n}$.

Finally, we can say that the roots of $\tilde{S}(n)$ have a certain degree (for the \mathbb{Z}_2 -grading or the \mathbb{Z}_n -grading), they are essential or non-essential, much like we did for $W(n)$ and $S(n)$. \diamond

Case $\mathfrak{g} = H(n)$: Let $\mathfrak{g} := H(n)$. Now $\mathfrak{g}_0 \cong \mathfrak{so}_n$, we fix the Cartan subalgebra \mathfrak{h} as above, and then $\bar{\mathfrak{h}} := \mathfrak{h} + \mathbb{K}\mathcal{E} \supsetneq \mathfrak{h}$. To describe the root system and the root spaces in this case, we point out the explicit form of the action of \mathcal{E} on the basis vectors of $\mathfrak{g} := H(n)$ considered in §2.16. Let $D_{\underline{\xi}^a}$ be any one of these elements, with $\underline{a} \in \{0, 1\}^n$, with $0 < |\underline{a}| < n$. The formulas in §2.11 give

$$[\mathcal{E}, D_{\underline{\xi}^a}] = (|\underline{a}| - 2) D_{\underline{\xi}^a} \quad \forall \underline{a} \in \{0, 1\}^n : 0 < |\underline{a}| < n \quad (2.26)$$

As before, we write $n = 2r$ or $n = 2r + 1$, with $r := \lfloor n/2 \rfloor$. Let now $\{\varepsilon_1, \dots, \varepsilon_r\}$ be the standard basis in the weight space of $\mathfrak{g}_0 = \mathfrak{so}_n$: adding δ we get a basis of $\bar{\mathfrak{h}}^*$. The root system is

$$\begin{aligned} \Delta &= \{ \pm \varepsilon_{i_1} \pm \dots \pm \varepsilon_{i_k} + m\delta \mid 1 \leq i_1 < \dots < i_k \leq r, k-2 \leq m \leq n-2, m \geq -1, m-k \in 2\mathbb{Z} \} \quad \text{if } n = 2r \\ \Delta &= \{ \pm \varepsilon_{i_1} \pm \dots \pm \varepsilon_{i_k} + m\delta \mid 1 \leq i_1 < \dots < i_k \leq r, k-2 \leq m \leq n-2, m \geq -1 \} \quad \text{if } n = 2r+1 \end{aligned}$$

As to root spaces, consider any root $\alpha = \pm \varepsilon_{i_1} \pm \dots \pm \varepsilon_{i_k} + m\delta$ written as $\alpha = \sum_{j=1}^r d_j \varepsilon_j + m\delta$ with $d_j \in \{+1, 0, -1\}$ for all j . Then the formulas in §2.17 along with (2.26) lead to find that for every root $\alpha = \sum_{j=1}^r d_j \varepsilon_j + m\delta$ the root space $\mathfrak{g}_\alpha = \sum_{j=1}^r d_j \varepsilon_j + m\delta$ has \mathbb{K} -basis the set

$$\{ D_{\underline{\xi}^a} \mid \underline{a} \in \{0, 1\}^n : |\underline{a}| - 2 = m, \underline{a}(j) - \underline{a}(r+j) = d_j \quad \forall j \}$$

The height of $\alpha = \pm \varepsilon_{i_1} \pm \dots \pm \varepsilon_{i_k} + m\delta$ is $ht(\alpha) = m$; in particular, if $D_{\underline{\xi}^a} \in \mathfrak{g}_\alpha$ then $|\underline{a}| = ht(\alpha) + 2$; the parity of α is the same as m , and its multiplicity is $\mu(\alpha) = \binom{r}{\lfloor (m-k)/2 \rfloor}$, where $\lfloor (m-k)/2 \rfloor$ is the integral part of $(m-k)/2$.

Finally, note that the roots $\alpha = m\delta$ are the only ones whose double might be a root too. \diamond

2.19. Finiteness properties of roots and root vector action. As we saw in §2.18, the root space decomposition yields a Q -grading of \mathfrak{g} . As a consequence, each root vector acts nilpotently. Actually, we can make this result more precise. We begin with some easy properties of roots:

Lemma 2.20. *Let $\alpha, \beta \in \Delta$. Then $(t\alpha + \beta) \notin \Delta$ for all $t > 2$. Moreover, if $2\alpha \in \Delta$ or $3\alpha \in \Delta$, then $\mathfrak{g} = H(n)$, $\alpha \in \{m\delta \mid m \in \mathbb{Z}\}$, or (only for the first case) $\mathfrak{g} = \tilde{S}(n)$, $\alpha \in \{-\varepsilon_i\}_{i=1, \dots, n}$.*

Here now are the finiteness properties of the root vector action we need:

Proposition 2.21. *The following hold:*

- (a) *Let $\alpha \in \Delta$, $\alpha \neq -\varepsilon_i$ if $\mathfrak{g} = \tilde{S}(n)$ or $\alpha \neq (2N+1)\delta$ if $\mathfrak{g} = H(2r+1)$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \{0\}$.*
- (b) *Let $\alpha \in \Delta_0$, $x_\alpha \in \mathfrak{g}_\alpha$. Then $ad(x_\alpha)^3 = 0$; if $\alpha \notin \Delta_0$, then $ad(x_\alpha)^2 = 0$.*

Proof. (a) If \mathfrak{g} is of type W , S or \tilde{S} , then direct inspection shows $2\alpha \notin \Delta$ for $\alpha \in \Delta$, with $\alpha \neq -\varepsilon_i$ in type \tilde{S} . If instead \mathfrak{g} is of type $H(n)$ then $2\alpha \notin \Delta$ whenever $\alpha \notin \mathbb{Z}\delta$ (as direct inspection shows). Then in all these cases one has $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \subseteq \mathfrak{g}_{2\alpha} = \{0\}$. Finally, assume \mathfrak{g} is of type $H(n)$ and $\alpha \in \mathbb{Z}\delta$, say $\alpha = k\delta$. The root space \mathfrak{g}_α has \mathbb{K} -basis the set $\{ D_{\underline{\xi}^a} \mid \underline{a}(s) = \underline{a}(r+s) \quad \forall 1 \leq s \leq r \}$, and by formulas (2.19–21) we may have $[D_{\underline{\xi}^a}, D_{\underline{\xi}^b}] = D_{\{\underline{\xi}^a, \underline{\xi}^b\}} \neq 0$ only if n is odd ($= 2r+1$).

(b) First of all, assume $\alpha \in \Delta_0 \setminus \Delta_0$, and consider any other root $\beta \in \Delta$. By Lemma 2.20 one has $3\alpha \notin \Delta$ and $3\alpha + \beta \notin \Delta$: this easily implies $ad(x_\alpha)^3 = 0$.

Now let \mathfrak{g} be of type W , S or \tilde{S} . Direct inspection shows that, given $\alpha \in \Delta_0$ and $\beta \in \Delta$, one has $2\alpha + \beta \in \Delta$ only if $\alpha \in \Delta_0$. Then one can argue as above for the action of $ad(x_\alpha)^2$ onto \mathfrak{h} and onto root spaces \mathfrak{g}_β , and eventually prove the second part of the claim for types W , S and \tilde{S} .

Let now \mathfrak{g} be of type H , say $\mathfrak{g} = H(n)$. From §2.18 we know that the root vector x_α is a linear combination of root vectors of the form $D_{\underline{\xi}^a}$, say $x_\alpha = \sum_k c_k D_{\underline{\xi}^{a_k}}$ for some $c_k \in \mathbb{K}$.

First assume $\alpha \notin \mathbb{Z}\delta$. Then $2\alpha \notin \Delta$, by Lemma 2.20, thus $\mathfrak{g}_{2\alpha} = \{0\}$ and for any pair of summands in the expansion of x_α we have $\{ D_{\underline{\xi}^{a_{k'}}}, D_{\underline{\xi}^{a_{k''}}} \} \in \{ \mathfrak{g}_\alpha, \mathfrak{g}_\alpha \} \subseteq \mathfrak{g}_{2\alpha} = \{0\}$.

Now let $\alpha \in \mathbb{Z}\delta$, say $\alpha = m\delta$. Then m is even because $\alpha \in \Delta_{\bar{0}}$; but $|\underline{a}_k| = ht(\alpha) + 2 = m + 2$ for each \underline{a}_k occurring in the expansion of x_α , so in particular $|\underline{a}_k|$ is (independent of k and) even, thus in the end \underline{a}_k has the property that $\underline{a}_k(s) = \underline{a}_k(\ell + s)$ for all $1 \leq s \leq \ell$. Now (2.22) applies, yielding $\{\underline{\xi}^{\underline{a}_{k'}}, \underline{\xi}^{\underline{a}_{k''}}\} = 0$ and so in the end $\{D_{\underline{\xi}^{\underline{a}_{k'}}}, D_{\underline{\xi}^{\underline{a}_{k''}}}\} = D_{\{\underline{\xi}^{\underline{a}_{k'}}, \underline{\xi}^{\underline{a}_{k''}}\}} = D_0 = 0$.

In any case, we found that all summands in the expansion of x_α do commute with each other. It follows that $ad(x_\alpha)^2 = \sum_k c_k^2 ad(D_{\underline{\xi}^{\underline{a}_k}})^2$; so it is enough to prove the claim for $x_\alpha = D_{\underline{\xi}^{\underline{a}}}$.

Again, $x_\alpha = D_{\underline{\xi}^{\underline{a}}}$ implies $|\underline{a}| = ht(\alpha) + 2$: as $\alpha \in \Delta_{\bar{0}} \setminus \Delta_0$ implies $ht(\alpha) \geq 2$, this yields $|\underline{a}| \geq 4$. Then (2.23) gives $ad(\underline{\xi}^{\underline{a}})^2(\underline{\xi}^{\underline{b}}) = \{\underline{\xi}^{\underline{a}}, \{\underline{\xi}^{\underline{a}}, \underline{\xi}^{\underline{b}}\}\} = 0$ for $\underline{b} \in \{0, 1\}^n$: thus $ad(D_{\underline{\xi}^{\underline{a}}})^2(\mathfrak{g}_\beta) = \{0\}$ for all $\beta \in \Delta$. Similarly $ad(D_{\underline{\xi}^{\underline{a}}})^2(h) = [D_{\underline{\xi}^{\underline{a}}}, [D_{\underline{\xi}^{\underline{a}}}, h]] = -\alpha(h) [D_{\underline{\xi}^{\underline{a}}}, D_{\underline{\xi}^{\underline{a}}}] = 0$ for any $h \in \mathfrak{h}$, as $D_{\underline{\xi}^{\underline{a}}} \in \mathfrak{g}_{\bar{0}}$: thus $ad(D_{\underline{\xi}^{\underline{a}}})^2(\mathfrak{h}) = \{0\}$ too. Then arguing as above we get $ad(D_{\underline{\xi}^{\underline{a}}})^2 = 0$. \square

2.22. Triangular decompositions, Borel subalgebras and special sub-objects. Since $char(\mathbb{K}) = 0$, we can identify hereafter the fundamental subfield of \mathbb{K} with \mathbb{Q} .

Let $\mathfrak{h}_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \{H \in \mathfrak{h} \mid \alpha(H) \in \mathbb{Z}, \forall \alpha \in \Delta\}$; one sees easily that $\mathfrak{h}_{\mathbb{Q}}$ is an integral \mathbb{Q} -form of \mathfrak{h} , and $\alpha(h) \in \mathbb{Q}$ for all $h \in \mathfrak{h}_{\mathbb{Q}}$. We call $h \in \mathfrak{h}_{\mathbb{Q}}$ *regular* if $\alpha(h) \neq 0$ for all $\alpha \in \Delta$. Any regular $h \in \mathfrak{h}$ defines a partition $\Delta = \Delta^+ \amalg \Delta^-$ where $\Delta^+ := \{\alpha \in \Delta \mid \alpha(h) > 0\}$ and $\Delta^- := \{\alpha \in \Delta \mid \alpha(h) < 0\}$: the roots in Δ^+ are said to be *positive*, those in Δ^- *negative*. All this defines a *triangular decomposition* $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$ with $\mathfrak{g}^\pm := \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$, as well as *Borel subalgebras* $\mathfrak{b}^\pm := \mathfrak{h} \oplus \mathfrak{g}^\pm$. From now on, we fix a specific Borel subalgebra of \mathfrak{g}_0 , denoted \mathfrak{b}_0 , and we restrict ourselves to consider those Borel subalgebras of \mathfrak{g} containing \mathfrak{b}_0 . Among these, when $\mathfrak{g} \not\cong \tilde{S}(n)$ there is a *maximal* one, $\mathfrak{b}_{\max} := \mathfrak{b}_0 \oplus \left(\bigoplus_{i>0} \mathfrak{g}_i \right)$, and a *minimal* one, $\mathfrak{b}_{\min} := \mathfrak{b}_0 \oplus \mathfrak{g}_{-1}$.

For later use, we introduce notation (consistent with §2.18) $\Delta_{\bar{0}\uparrow} := \Delta_{\bar{0}} \setminus \Delta_0$, $\Delta_{\bar{1}\uparrow} := \Delta_{\bar{1}} \setminus \Delta_{-1}$, $\Delta_\bullet^\pm := \Delta_\bullet \cap \Delta^\pm$, $\tilde{\Delta}_\bullet^\pm := \tilde{\Delta}_\bullet \cap (\Delta_\bullet^\pm \times \mathbb{N}_+)$ where $\Delta_\bullet \in \{\Delta, \Delta_0, \Delta_{\bar{0}}, \Delta_{\bar{1}}, \Delta_{\bar{0}\uparrow}, \Delta_{\bar{1}\uparrow}\}$.

Starting from the root decomposition of §2.18, we can introduce special “sub-objects” of \mathfrak{g} :

Definition 2.23. Basing on the root decomposition in §2.18, we have $\mathfrak{g}_{\bar{0}} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_{\bar{0}}} \mathfrak{g}_\alpha \right)$, $\mathfrak{g}_{\bar{1}} = \bigoplus_{\gamma \in \Delta_{\bar{1}}} \mathfrak{g}_\gamma$ and $\mathfrak{g}_0 = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha \right)$. Then set $\mathfrak{g}_{\bar{0}\uparrow} := \bigoplus_{\alpha \in \Delta_{\bar{0}\uparrow}} \mathfrak{g}_\alpha$, $\mathfrak{g}_{\bar{1}\uparrow} := \bigoplus_{\gamma \in \Delta_{\bar{1}\uparrow}} \mathfrak{g}_\gamma$, and $\mathfrak{g}_{t\uparrow} = \bigoplus_{z>t} \mathfrak{g}_z$ for all $t \geq -1$. Note then that $\mathfrak{g}_{-1\uparrow} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}\uparrow}$ and $\mathfrak{g}_{0\uparrow} = \mathfrak{g}_{\bar{0}\uparrow} \oplus \mathfrak{g}_{\bar{1}\uparrow}$; note also that $(\mathfrak{g}_{t\uparrow})_{\bar{0}} = \mathfrak{g}_{\bar{0}} \cap \mathfrak{g}_{t\uparrow} = \bigoplus_{\substack{z>t \\ z \in 2\mathbb{Z}}} \mathfrak{g}_z$, for all $t \geq -1$. For $\mathfrak{g} \not\cong \tilde{S}(n)$ consider also $\mathfrak{g}_{-1,0} := \mathfrak{g}_{-1} \oplus \mathfrak{g}_{\bar{0}}$.

Remark 2.24. Note that \mathfrak{g}_0 , $\mathfrak{g}_{\bar{0}\uparrow}$ and $\mathfrak{g}_{\bar{0}}$ are Lie subalgebras of \mathfrak{g} , while $\mathfrak{g}_{\bar{0}\uparrow}$ is a Lie ideal of $\mathfrak{g}_{\bar{0}}$, and we have a Lie algebra splitting $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_{\bar{0}\uparrow}$ (semidirect product of Lie algebras), with \mathfrak{g}_0 reductive and $\mathfrak{g}_{\bar{0}\uparrow}$ nilpotent. Similarly, $(\mathfrak{g}_{t\uparrow})_{\bar{0}}$ is a Lie subalgebra of $\mathfrak{g}_{\bar{0}}$, for all $t \geq -1$; when $q > p$, $(\mathfrak{g}_{q\uparrow})_{\bar{0}}$ is a Lie ideal of $(\mathfrak{g}_{p\uparrow})_{\bar{0}}$. On the other hand, $\mathfrak{g}_{-1\uparrow}$ and $\mathfrak{g}_{0\uparrow}$ are Lie supersubalgebras of \mathfrak{g} , with $\mathfrak{g}_{-1\uparrow} \supseteq \mathfrak{g}_{0\uparrow}$. Moreover, $\mathfrak{g}_{0\uparrow}$ is a nilpotent Lie superalgebra, and a Lie ideal of $\mathfrak{g}_{-1\uparrow}$: then $\mathfrak{g}_{-1\uparrow} = \mathfrak{g}_0 \ltimes \mathfrak{g}_{0\uparrow}$ (semidirect product); similarly, for $\mathfrak{g} \not\cong \tilde{S}(n)$ we have $\mathfrak{g}_{-1,0} = \mathfrak{g}_{-1} \ltimes \mathfrak{g}_0$.

2.4 Basics on \mathfrak{g} -modules

Later on we shall work with \mathfrak{g} -modules and $\bar{\mathfrak{g}}$ -modules, so we specify now a few definitions.

Definition 2.25. Let \mathfrak{h} , resp. $\bar{\mathfrak{h}}$, be a fixed Cartan subalgebra of \mathfrak{g} , resp. of $\bar{\mathfrak{g}}$, as in §2.18.

(a) Any \mathfrak{g} -module V is said to be a *weight module* if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where we set $V_\lambda := \{v \in V \mid h.v = \lambda(h)v, \forall h \in \mathfrak{h}\}$ for all $\lambda \in \mathfrak{h}^*$. In this case, every V_λ is called a *weight space* of V , and every $\lambda \in \mathfrak{h}^*$ such that $V_\lambda \neq \{0\}$ is called a *weight* of V .

Let now V be a weight module (for \mathfrak{g}), and set $Supp(V) := \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq \{0\}\}$:

(b) for every $\lambda \in \mathfrak{h}^*$ we call *multiplicity* of λ the dimension $mult_V(\lambda) := \dim(V_\lambda)$;

(c) we call V *integrable* if all root vectors of \mathfrak{g} act locally nilpotently on it;

(d) if a splitting of roots into positive and negative ones has been fixed as in §2.22, we call *highest weight* of V any $\lambda \in \text{Supp}(V)$ such that $\mathfrak{g}_\alpha \cdot V_\lambda = \{0\}$, i.e. $\lambda + \alpha \notin \text{Supp}(V)$, for all $\alpha \in \Delta^+$; then we call *highest weight vector* (of weight λ) any $v_\lambda \in V_\lambda \setminus \{0\}$.

We adopt similar definitions and terminology when $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{h}}$ replace \mathfrak{g} and \mathfrak{h} respectively.

Also, notice that any \mathfrak{g} -module bears a structure of $\bar{\mathfrak{g}}$ -module too, with \mathcal{E} acting semisimply: moreover, if the \mathfrak{g} -module is in fact a weight module, then its structure of $\bar{\mathfrak{g}}$ -module can also be chosen so that it is still a weight module for $\bar{\mathfrak{g}}$ as well (all this is standard, see e.g. [12] or [16]).

Remarks 2.26.

(a) by standard theory of reductive Lie algebras, every finite dimensional \mathfrak{g} -module on which the element \mathcal{E} acts semisimply is automatically a weight module;

(b) if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ is any weight module, then $x_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$ for every root vector $x_\alpha \in \mathfrak{g}_\alpha$ of \mathfrak{g} ($\alpha \in \Delta$), by elementary calculations; it follows that every finite dimensional \mathfrak{g} -module — which is a weight module by (a) — is automatically integrable;

(c) \mathfrak{g} itself is an integrable weight $\bar{\mathfrak{g}}$ -module for the adjoint representation: the set of weights is $\Delta \cup \{0\}$, and weight spaces are the root spaces and \mathfrak{h} ; similarly (up to details) for $\bar{\mathfrak{g}}$ as a $\bar{\mathfrak{g}}$ -module.

(d) if $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ is any integrable (weight) module, then for each root vector $x_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta$) the formal infinite sum $\exp(x_\alpha) := \sum_{n \in \mathbb{N}} x_\alpha^n / n!$ gives a well-defined operator in $\text{GL}(V)$.

3 Integral structures

In this section we introduce the first, fundamental results we shall build upon to construct our “Chevalley type” supergroups associated with \mathfrak{g} . We keep notation and terminology as before.

3.1 Chevalley bases and Chevalley superalgebras

In this subsection we extend a classical result: the notion of “Chevalley basis” for (semi)simple Lie algebras. A similar notion was introduced in [9] for simple Lie superalgebras of *classical* type, and used to construct affine algebraic supergroups. We now do the same for the Cartan type case.

Definition 3.1. We call $r := \text{rk}(\mathfrak{g})$ the *rank* of \mathfrak{g} : by definition, it is the rank of the reductive Lie algebra \mathfrak{g}_0 , so $\text{rk}(W(n)) = n$, $\text{rk}(S(n)) = \text{rk}(\tilde{S}(n)) = n - 1$ and $\text{rk}(H(n)) = [n/2]$.

We call *Chevalley basis* of \mathfrak{g} any \mathbb{K} -basis $B = \{H_i\}_{i=1, \dots, r}; \coprod \left(\coprod_{\alpha \in \Delta} \{X_{\alpha, k}\}_{k=1, \dots, \mu(\alpha)} \right) = \{H_i\}_{i=1, \dots, r}; \coprod \{X_{\bar{\alpha}}\}_{\bar{\alpha} \in \tilde{\Delta}}$ of \mathfrak{g} which is *homogeneous* (for the cyclic grading of \mathfrak{g} , cf. subsec. 2.3) and enjoying the following properties (with notation of §2.18 for coroots):

(a) $\{H_1, \dots, H_r\}$ is a \mathbb{K} -basis of \mathfrak{h} , such that $\beta(H_1), \dots, \beta(H_r) \in \mathbb{Z}$ for all $\beta \in \Delta$ and $H_\alpha \in \mathfrak{h}_{\mathbb{Z}} := \text{Span}_{\mathbb{Z}}(H_1, \dots, H_r)$ for all $\alpha \in \Delta_0$; in particular, $[H_i, H_j] = 0$ for all $1 \leq i, j \leq r$;

(b) $\{X_{\alpha, k}\}_{k=1, \dots, \mu(\alpha)}$ is a \mathbb{K} -basis of \mathfrak{g}_α , for all $\alpha \in \Delta$; thus $[H_i, X_{\alpha, k}] = \alpha(H_i) X_{\alpha, k} \ \forall i, k$;

(c) $[X_{\alpha, k}, X_{\alpha, k}] = 0 \quad \forall \alpha \in \Delta_{\bar{0}}, k \in \{1, \dots, \mu(\alpha)\}$;

(d) $[X_{\alpha, 1}, X_{-\alpha, 1}] = H_\alpha \quad \forall \alpha \in \Delta_0$,

$[X_{\gamma, 1}, X_{-\gamma, k}] = \pm H_{\sigma_\gamma(k)} \quad \forall \gamma \in \Delta_{-1}, k = 1, \dots, \mu(-\gamma)$, for some embeddings $\sigma_\gamma: \{1, \dots, \mu(-\gamma)\} \hookrightarrow \{1, \dots, r\}$ such that $\{\pm H_{\sigma_\gamma(k)}\}_{\gamma \in \Delta_{-1}, k=1, \dots, \mu(-\gamma)} = \{\pm H_1, \dots, \pm H_r\}$;

(e) $[X_{\alpha, k}, X_{\beta, h}] = 0 \quad \forall \alpha, \beta \in \Delta: \alpha + \beta \notin (\Delta \cup \{0\})$;

(f) $[X_{\alpha, k}, X_{\beta, h}] = \sum_{t=1}^{\mu(\alpha+\beta)} c_{\alpha, k}^{\beta, h}(t) X_{\alpha+\beta, t} \quad \forall \alpha, \beta \in \Delta: \alpha + \beta \in \Delta$,

for some $c_{\alpha, k}^{\beta, h}(t) \in \{0, \pm 1, \pm 2\}$ such that

(f.1) if $\alpha, \beta \in \Delta_0$ then $c_{\alpha,1}^{\beta,1}(1) = \pm(p+1)$ where $p \in \mathbb{N}$ is such that $\beta - p\alpha \in \Delta$ and $\beta - (p+1)\alpha \notin \Delta$ (note that in this case $\mu(\alpha) = \mu(\beta) = \mu(\alpha + \beta) = 1$),

(f.2) if $(\alpha, \beta) \in (\Delta_{-1} \times \Delta) \cup (\Delta \times \Delta_{-1})$, then there is one and only one index t' such that $c_{\alpha,k}^{\beta,h}(t') = \pm 1$, $c_{\alpha,k}^{\beta,h}(t) = 0 \quad \forall t' \neq t$,

(f.3) if $\alpha = \beta$ (hence $2\alpha \in \Delta$; this occurs only for $\mathfrak{g} = \tilde{S}(n)$, $\alpha = -\epsilon_i$, or $\mathfrak{g} = H(2r+1)$, $\alpha = m\delta$ with m odd), there is at most one t' such that $c_{\alpha,k}^{\beta,h}(t') \neq 0$, and then $c_{\alpha,k}^{\beta,h}(t') = \pm 2$;

(g) if $\alpha \in \Delta_0$ and $2\alpha + \beta \in \Delta$, there exists a unique $t' \in \{1, \dots, \mu(2\alpha + \beta)\}$ such that

$$[X_{\alpha,1}, [X_{\alpha,1}, X_{\beta,h}]] = \pm 2 X_{2\alpha+\beta,t'};$$

(h) for all $\gamma \in \Delta_1$, we have (with notation as in (f.3.) above)

$$X_{\gamma,k}^{(2)} = 0 \quad \text{if all } c_{\gamma,k}^{\gamma,k}(t) \text{'s are zero,} \quad \text{and} \quad X_{\gamma,k}^{(2)} = 2^{-1} c_{\gamma,k}^{\gamma,k}(t') X_{2\gamma,t'} \quad \text{otherwise.}$$

If B is any Chevalley basis of \mathfrak{g} , we set $\mathfrak{g}^{\mathbb{Z}} := \mathbb{Z}$ -span of B , and we call it *Chevalley superalgebra*.

Remarks 3.2.

(a) The above definition extends to the Lie superalgebra \mathfrak{g} the notion of Chevalley basis for (semi)simple Lie algebras: in particular, if B is a Chevalley basis of \mathfrak{g} then $B \cap \mathfrak{g}_0$ is a Chevalley basis — in a standard sense, extended to reductive Lie algebras in case W — of \mathfrak{g}_0 .

In the present formulation, the conditions we give are clearly redundant, and may be simplified.

(b) By its very definition, the Chevalley superalgebra $\mathfrak{g}^{\mathbb{Z}}$ is a Lie superalgebra over \mathbb{Z} .

(c) When \mathfrak{g} is not of type W , so that $\mathfrak{g} \subsetneq \overline{\mathfrak{g}}$, if B is any Chevalley basis for \mathfrak{g} we can as well consider $\overline{B} := B \amalg \{H_\delta\}$: this plays the role of a “Chevalley basis” for $\overline{\mathfrak{g}}$, and we can develop all the theory which follows hereafter with $\overline{\mathfrak{g}}$ and $\overline{\mathfrak{h}}$ replacing \mathfrak{g} and \mathfrak{h} respectively.

(d) For notational convenience, in the following I shall also use the notation $X_{\eta,k} := 0$ when $k \in \mathbb{N}_+$ and η belongs to Q (the \mathbb{Z} -span of Δ) but $\eta \notin \Delta$.

We prove the existence of Chevalley bases of \mathfrak{g} by providing explicit examples, as follows:

Examples 3.3. Explicit examples of Chevalley bases.

(a) Case $W(n)$, first example: Let $\mathfrak{g} := W(n)$, and take the subset

$$B' \equiv B_{W(n)} := \{H_i\}_{i=1,\dots,n}; \amalg \left(\amalg_{\alpha \in \Delta} \{X_{\alpha,k}\}_{k=1,\dots,\mu(\alpha)} \right)$$

considered in §2.10, but now written in different notation, namely

$$\begin{aligned} H_i &:= \xi_i \partial_i && \text{for all } i = 1, \dots, r (= n), \\ X_{\alpha,1} &:= \xi_{i_1} \cdots \xi_{i_s} \partial_j && \text{for every root of the form } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s} - \varepsilon_j \quad (j \notin \{i_1, \dots, i_s\}), \\ X_{\alpha,k} &:= \xi_{i_1} \cdots \xi_{i_s} \xi_{j_k} \partial_{j_k} && \text{for every root of the form } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s}, \text{ where } j_k \text{ is the} \\ &&& k\text{-th index in } \{1, \dots, n\} \setminus \{i_1, \dots, i_s\}, \text{ for } k = 1, \dots, \mu(\alpha). \end{aligned}$$

By the results in §2.11 and §2.18 one checks by direct analysis that B' is a Chevalley basis.

Indeed, this specific Chevalley basis has even stronger properties than the prescribed ones. Namely, the $c_{\alpha,k}^{\beta,h}(t)$ occurring in part (f) of Definition 3.1 satisfy, besides (f.1) and (f.2), the following (stronger) properties: $c_{\alpha,k}^{\beta,h}(t) \in \{0, \pm 1\}$, and there exist at most two indices t_1 and t_2 such that $c_{\alpha,k}^{\beta,h}(t_1) = \pm 1$, $c_{\alpha,k}^{\beta,h}(t_2) = \pm 1$, and $c_{\alpha,k}^{\beta,h}(t') = 0$ for all $t' \notin \{t_1, t_2\}$.

(b) Case $S(n)$: Let $\mathfrak{g} := S(n)$, and take the subset

$$B \equiv B_{S(n)} := \{H_i\}_{i=1,\dots,n-1}; \amalg \left(\amalg_{\alpha \in \Delta} \{X_{\alpha,k}\}_{k=1,\dots,\mu(\alpha)} \right)$$

considered in §2.12, but now written with another notation, namely

$$H_i := \xi_i \partial_i - \xi_{i+1} \partial_{i+1} \quad \text{for all } i = 1, \dots, r (= n-1),$$

$$\begin{aligned}
X_{\alpha,1} &:= \xi_{i_1} \cdots \xi_{i_s} \partial_j \quad \text{for every root of the form } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s} - \varepsilon_j, \ j \notin \{i_1, \dots, i_s\}, \\
X_{\alpha,k} &:= \xi_{i_1} \cdots \xi_{i_s} (\xi_{j_k} \partial_{j_k} - \xi_{j_{k+1}} \partial_{j_{k+1}}) \quad \text{for every root of the form } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s}, \text{ where} \\
&\quad \{j_1, \dots, j_{\mu(\alpha)}, j_{\mu(\alpha)+1}\} := \{1, \dots, n\} \setminus \{i_1, \dots, i_s\} \text{ with } j_1 < \cdots < j_{\mu(\alpha)} < j_{\mu(\alpha)+1}.
\end{aligned}$$

Here again, direct analysis based on §2.13 and §2.18 shows that B is a Chevalley basis of $S(n)$.

(c) Case $\tilde{S}(n)$: Let $\mathfrak{g} := \tilde{S}(n)$, and take the subset

$$B \equiv B_{\tilde{S}(n)} := \{H_i\}_{i=1, \dots, n-1}; \coprod \left(\coprod_{\alpha \in \Delta} \{X_{\alpha,k}\}_{k=1, \dots, \mu(\alpha)} \right)$$

introduced in §2.14 and now written with a different notation, namely

$$\begin{aligned}
H_i &:= \xi_i \partial_i - \xi_{i+1} \partial_{i+1} && \text{for all } i = 1, \dots, r (= n-1), \\
X_{\alpha,1} &:= \xi_{i_1} \cdots \xi_{i_s} \partial_j \quad \text{for each root of the form } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s} - \varepsilon_j, \ j \notin \{i_1, \dots, i_s\}, \ s > 0, \\
X_{\alpha,1} &:= (\xi_1 \cdots \xi_n - 1) \partial_j && \text{for every root of the form } \alpha = -\varepsilon_j \ (j = 1, \dots, n), \\
X_{\alpha,k} &:= \xi_{i_1} \cdots \xi_{i_s} (\xi_{j_k} \partial_{j_k} - \xi_{j_{k+1}} \partial_{j_{k+1}}) \quad \text{for every root of the form } \alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_s}, \text{ where} \\
&\quad \{j_1, \dots, j_{\mu(\alpha)}, j_{\mu(\alpha)+1}\} := \{1, \dots, n\} \setminus \{i_1, \dots, i_s\} \text{ with } j_1 < \cdots < j_{\mu(\alpha)} < j_{\mu(\alpha)+1}.
\end{aligned}$$

Again, direct analysis (via §2.15 §2.18) shows that this B is indeed a Chevalley basis of $\tilde{S}(n)$.

(d) Case $W(n)$, second example: Let again $\mathfrak{g} := W(n)$. For any $i \in \{1, \dots, n\}$ and any $j_\alpha \in \{1, \dots, \mu(\alpha)\}$ for $\{\alpha \in \Delta : ht(\alpha) \geq 1\}$ — $\mu(\alpha)$ being the multiplicity in $W(n)$ — consider

$$B'' := B_{S(n)} \coprod \{\xi_i \partial_i\} \coprod \{X_{\alpha,j_\alpha} := \xi_{a_1} \cdots \xi_{a_s} \xi_{j_\alpha} \partial_{j_\alpha} \mid \alpha \in \Delta : ht(\alpha) > 1\}$$

where we wrote $\alpha = \varepsilon_{a_1} + \cdots + \varepsilon_{a_s}$ for every root α with $ht(\alpha) > 1$ (so that the string (a_1, \dots, a_s) depends on α itself). Yet another direct check shows that B'' is a Chevalley basis of $W(n)$.

(e) Case $H(n)$: Let $\mathfrak{g} := H(n)$, and $B_{H(n)} := \{D_{\underline{\varepsilon}} \mid \underline{\varepsilon} \in \{0, 1\}^n, 0 < |\underline{\varepsilon}| < n\}$. Set

$$H_i := -D_{\xi_i \xi_{r+i}} = \xi_i \partial_i - \xi_{r+i} \partial_{r+i} \quad \text{for all } i = 1, \dots, r (= [n/2]);$$

then for any root $\alpha = \varepsilon_{i_1} + \cdots + \varepsilon_{i_p} - \varepsilon_{j_1} - \cdots - \varepsilon_{j_q} + m\delta$ set $s := [(m-p-q)/2]$ and pick the root vector (with ϕ as in §2.16)

$$\begin{aligned}
X_{\alpha,k} &:= \pm D_{\xi_{i_1} \cdots \xi_{i_p} \xi_{r+j_1} \cdots \xi_{r+j_q} \xi_{t_1} \cdots \xi_{t_s} \xi_{r+t_1} \cdots \xi_{r+t_s}} && \text{if } (m-p-q) \text{ is even,} \\
X_{\alpha,k} &:= \pm \sqrt{2} D_{\xi_{i_1} \cdots \xi_{i_p} \xi_{r+j_1} \cdots \xi_{r+j_q} \xi_{t_1} \cdots \xi_{t_s} \xi_{r+t_1} \cdots \xi_{r+t_s} \xi_{2r+1}} && \text{if } (m-p-q) \text{ is odd,}
\end{aligned}$$

for every choice of $t_1, \dots, t_s \in \{1, \dots, r\} \setminus (\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\})$ with $t_1 < \cdots < t_s$, where $k \in \{1, \dots, \mu(\alpha)\}$ is used to order the possible choices of ordered subset of indices $\{t_1, \dots, t_s\}$.

N.B.: the root vectors of second type have to be considered only when n itself is odd.

Now, using the formulas and results in §2.17 and §2.18, one checks that the set of all H_i 's and all $X_{\alpha,k}$'s defined above (for suitable choice of signs) is indeed a Chevalley basis of $H(n)$. \diamond

3.2 The Kostant superalgebra

For any \mathbb{K} -algebra A , given $m \in \mathbb{N}$ and $y \in A$ we define the m -th binomial coefficient $\binom{y}{m}$ and the m -th divided power $y^{(m)}$ by $\binom{y}{m} := \frac{y(y-1)\cdots(y-m+1)}{m!}$, $y^{(m)} := y^m/m!$.

We start with a (standard) classical result, concerning \mathbb{Z} -valued polynomials:

Lemma 3.4. (cf. [13], §26.1) Let $\mathbb{K}[\underline{y}] := \mathbb{K}[y_1, \dots, y_t]$ be the \mathbb{K} -algebra of polynomials in y_1, \dots, y_t , and $\text{Int}_{\mathbb{Z}}(\mathbb{K}[\underline{y}]) := \{f \in \mathbb{K}[\underline{y}] \mid f(z_1, \dots, z_t) \in \mathbb{Z} \ \forall z_1, \dots, z_t \in \mathbb{Z}\}$. Then $\text{Int}_{\mathbb{Z}}(\mathbb{K}[\underline{y}])$ is a \mathbb{Z} -subalgebra of $\mathbb{K}[\underline{y}]$, free as a \mathbb{Z} -(sub)module, with basis $\{\prod_{i=1}^t \binom{y_i}{n_i} \mid n_1, \dots, n_t \in \mathbb{N}\}$.

Let $U(\mathfrak{g})$ be the universal enveloping superalgebra of \mathfrak{g} . We recall that this can be realized as the quotient of the tensor superalgebra $T(\mathfrak{g})$ by the two-sided homogeneous ideal generated by $\left\{ x \otimes y - (-1)^{p(x)p(y)} y \otimes x - [x, y], z \otimes z - z^{(2)} \mid x, y \in \mathfrak{g}_{\bar{0}} \cup \mathfrak{g}_{\bar{1}}, z \in \mathfrak{g}_{\bar{1}} \right\}$.

Fix a Chevalley basis $B = \{H_i\}_{i=1, \dots, r}; \coprod \{X_{\alpha, k}\}_{\alpha \in \tilde{\Delta}, k=1, \dots, \mu(\alpha)} = \{H_i\}_{i=1, \dots, r}; \coprod \{X_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \tilde{\Delta}}$ of \mathfrak{g} as in Definition 3.1, and let $\mathfrak{h}_{\mathbb{Z}}$ be the free \mathbb{Z} -module with basis $\{H_1, \dots, H_r\}$. For $h \in \mathfrak{h}_{\mathbb{Z}}$, we denote by $h(H_1, \dots, H_r)$ the expression of h as a function of the H_i 's. From Lemma 3.4 we have:

Corollary 3.5. $\mathbb{H}_{\mathbb{Z}} := \{h \in U(\mathfrak{h}) \mid h(z_1, \dots, z_r) \in \mathbb{Z}, \forall z_1, \dots, z_r \in \mathbb{Z}\}$ is a free \mathbb{Z} -submodule of $U(\mathfrak{h})$, with basis $B_{U(\mathfrak{h})} := \left\{ \prod_{i=1}^r \binom{H_i}{m_i} \mid m_1, \dots, m_r \in \mathbb{N} \right\}$. Moreover, it coincides with the \mathbb{Z} -subalgebra of $U(\mathfrak{g})$ generated by all the elements $\binom{H-z}{m}$ with $H \in \mathfrak{h}_{\mathbb{Z}}, z \in \mathbb{Z}, m \in \mathbb{N}$.

We are now ready to define the Kostant superalgebra. Like in [9], we mimic the classical construction, but making a suitable distinction between the roles of even and odd root vectors.

Definition 3.6. We call *Kostant superalgebra* of $U(\mathfrak{g})$ the unital \mathbb{Z} -subsuperalgebra $K_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$ generated by all elements $\binom{H_i}{m}, X_{\tilde{\alpha}}^{(m)}, X_{\tilde{\gamma}}$ for $m \in \mathbb{N}, 1 \leq i \leq r, \tilde{\alpha} \in \tilde{\Delta}_{\bar{0}}, \tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}$.

Remarks 3.7. (a) The classical notion — suitably adapted to the reductive Lie algebra \mathfrak{gl}_n when $\mathfrak{g} = W(n)$ — defines the Kostant's \mathbb{Z} -form of $U(\mathfrak{g}_0)$, call it $K_{\mathbb{Z}}(\mathfrak{g}_0)$, as the unital \mathbb{Z} -subalgebra of $U(\mathfrak{g}_0)$ generated by the elements $X_{\tilde{\alpha}}^{(m)}, \binom{H_i}{m}$ with $\tilde{\alpha} \in \tilde{\Delta}_0$ and $m \in \mathbb{N}$. Then $K_{\mathbb{Z}}(\mathfrak{g}) \supseteq K_{\mathbb{Z}}(\mathfrak{g}_0)$.

(b) As a matter of notation, we shall always read $X_{\tilde{\alpha}}^{(m)} := \delta_{m,0}$ if $\tilde{\alpha} \notin \tilde{\Delta}$, for any $m \in \mathbb{N}$.

We shall use the following result, proved by induction (cf. also [13], §26.2, for part (a)):

Lemma 3.8. Let \mathfrak{l} be a Lie \mathbb{K} -algebra, and $\ell, m \in \mathbb{N}, \ell \wedge m := \min(\ell, m)$.

(a) Let $E, F \in \mathfrak{l}, H := [E, F] \in \mathfrak{l}$, and assume that $[H, E] = +2E, [H, F] = -2F$. Then

$$E^{(\ell)} F^{(m)} = \sum_{s=0}^{\ell \wedge m} F^{(m-s)} \binom{H-m-\ell+2s}{s} E^{(\ell-s)} \quad \text{inside } U(\mathfrak{l}).$$

(b) Let $A, B \in \mathfrak{l}, C := [A, B] \in \mathfrak{l}$, and assume also that $[A, C] = 0, [B, C] = 0$. Then

$$A^{(\ell)} B^{(m)} = \sum_{q=0}^{\ell \wedge m} B^{(m-q)} C^{(q)} A^{(\ell-q)} \quad \text{inside } U(\mathfrak{l}).$$

(c) Let $L, M \in \mathfrak{l}, N := [L, M], 2T := [L, N] \in \mathfrak{l}$. Assume also that $[M, N] = [L, T] = 0$ (then $[M, T] = [N, T] = 0$ as well). Then

$$L^{(\ell)} M^{(m)} = \sum_{s=0}^{\ell \wedge m} M^{(m-s)} \sum_{t+q=s} T^{(t)} N^{(q)} L^{(\ell-2t-q)} \quad \text{inside } U(\mathfrak{l}).$$

(d) Let $X, Y \in \mathfrak{l}$, and assume that $[X, Y] = 0$. Then

$$(X + Y)^{(m)} = \sum_{u=0}^m X^{(m-u)} Y^{(u)} = \sum_{v=0}^m Y^{(m-v)} X^{(v)} \quad \text{inside } U(\mathfrak{l}).$$

3.3 Commutation rules and Kostant's PBW theorem

In the classical setup, a description of $K_{\mathbb{Z}}(\mathfrak{g}_0)$ comes from a “PBW-like” theorem: namely, $K_{\mathbb{Z}}(\mathfrak{g}_0)$ is a free \mathbb{Z} -module with \mathbb{Z} -basis the set of ordered monomials (w. r. to any total order) whose factors are divided powers in the $X_{\tilde{\alpha}} (\tilde{\alpha} \in \tilde{\Delta}_0)$ or binomial coefficients in the $H_i (i = 1, \dots, n)$.

We shall prove a similar result for \mathfrak{g} , our Lie superalgebra of Cartan type. Like for (semi)simple Lie algebras — and also for simple Lie superalgebras of *classical* type, cf. [9], §4 — this follows from a direct analysis of commutation rules among the generators of $K_{\mathbb{Z}}(\mathfrak{g})$. To this end, we list hereafter all such rules, and also some slightly more general relations. We split the list into two sections: (1) relations involving only even generators; (2) relations involving also odd generators.

The relevant feature is that all coefficients in these relations are in \mathbb{Z} .

(1) **Even generators only** (that is $\binom{H_i}{m}$'s and $X_{\tilde{\alpha}}^{(m)}$'s only, $\tilde{\alpha} \in \tilde{\Delta}_0$):

$$\binom{H_i}{\ell} \binom{H_j}{m} = \binom{H_j}{m} \binom{H_i}{\ell} \quad \forall i, j \in \{1, \dots, r\}, \quad \forall \ell, m \in \mathbb{N} \quad (3.1)$$

$$X_{\tilde{\alpha}}^{(m)} f(H) = f(H - m \pi(\tilde{\alpha})(H)) X_{\tilde{\alpha}}^{(m)} \quad \forall \tilde{\alpha} \in \tilde{\Delta}_0, H \in \mathfrak{h}, m \in \mathbb{N}, f(T) \in \mathbb{K}[T] \quad (3.2)$$

$$X_{\tilde{\alpha}}^{(\ell)} X_{\tilde{\alpha}}^{(m)} = \binom{\ell+m}{m} X_{\tilde{\alpha}}^{(\ell+m)} \quad \forall \tilde{\alpha} \in \tilde{\Delta}_0, \quad \forall \ell, m \in \mathbb{N} \quad (3.3)$$

$$X_{\tilde{\alpha}}^{(\ell)} X_{\tilde{\beta}}^{(m)} = X_{\tilde{\beta}}^{(m)} X_{\tilde{\alpha}}^{(\ell)} \quad \forall \tilde{\alpha}, \tilde{\beta} \in \tilde{\Delta}_0 : \pi(\tilde{\alpha}) + \pi(\tilde{\beta}) \notin (\Delta \cup \{0\}), \quad \forall \ell, m \in \mathbb{N} \quad (3.4)$$

$$X_{\alpha,1}^{(m)} X_{-\alpha,1}^{(\ell)} = \sum_{s=0}^{\ell \wedge m} X_{-\alpha,1}^{(\ell-s)} \binom{H_{\alpha} - m - \ell + 2s}{s} X_{\alpha,1}^{(m-s)} \quad (3.5)$$

$\forall \alpha \in \Delta_0, \quad \forall \ell, m \in \mathbb{N}, \quad \text{with } \ell \wedge m := \min(\ell, m)$

$$X_{\alpha,k}^{(\ell)} X_{\beta,h}^{(m)} = \sum_{q=0}^{\ell \wedge m} X_{\beta,h}^{(m-q)} \left(\sum_{\sum q_t=q} \prod_{t=1}^{\mu(\alpha+\beta)} (c_{\alpha,k}^{\beta,h}(t))^{q_t} X_{\alpha+\beta,t}^{(q_t)} \right) X_{\alpha,k}^{(\ell-q)} \quad (3.6)$$

$$\forall \alpha \in \Delta_0, \beta \in \Delta_0 : \alpha + \beta \in \Delta, 2\alpha + \beta \notin \Delta, \alpha + 2\beta \notin \Delta, \quad \forall \ell, m \in \mathbb{N}$$

$$X_{\alpha,1}^{(\ell)} X_{\beta,h}^{(m)} = \sum_{s=0}^{\ell \wedge m} X_{\beta,h}^{(m-s)} \sum_{p+q=s} \epsilon^p X_{2\alpha+\beta,t'}^{(p)} \left(\sum_{\sum q_t=q} \prod_{t=1}^{\mu(\alpha+\beta)} (c_{\alpha,1}^{\beta,h}(t))^{q_t} X_{\alpha+\beta,t}^{(q_t)} \right) X_{\alpha,1}^{(\ell-2p-q)} \quad (3.7)$$

$$\forall \alpha \in \Delta_0, \beta \in \Delta_0 \setminus \Delta_0 : \alpha + \beta, 2\alpha + \beta \in \Delta, \quad \forall \ell, m \in \mathbb{N}$$

$$X_{\alpha,1}^{(\ell)} X_{\beta,1}^{(m)} = X_{\beta,1}^{(m)} X_{\alpha,1}^{(\ell)} + l.h.t. \quad \forall \alpha, \beta \in \Delta_0, \quad \forall \ell, m \in \mathbb{N} \quad (3.8)$$

where $c_{\alpha,1}^{\beta,h}(t)$ and $X_{\alpha+\beta,t}$ are as in Definition 3.1(f), while $\epsilon = \pm 1$ and the index t' are such that $[X_{\alpha,1}, [X_{\alpha,1}, X_{\beta,h}]] = \epsilon 2 X_{2\alpha+\beta,t'}$ as in Definition 3.1(g), and *l.h.t.* (= “lower height terms”) stands for a \mathbb{Z} -linear combinations of monomials in the $X_{\eta}^{(q)}$'s and in the $\binom{H_i}{c}$'s whose “height” — i.e., the sum of all “exponents” q occurring in such a monomial — is less than $\ell + m$.

Proof. Relations (3.1), (3.2), (3.3) and (3.5) hold by definitions, along with Lemma 3.8(a).

If $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Delta}$ and $\pi(\tilde{\alpha}) + \pi(\tilde{\beta}) \notin (\Delta \cup \{0\})$, then we get $[X_{\tilde{\alpha}}, X_{\tilde{\beta}}] = 0$ by Definition 3.1(e), so $X_{\tilde{\alpha}}$ and $X_{\tilde{\beta}}$ commute with each other: this implies (3.4).

Relations (3.6) follow as an application of Lemma 3.8(b) to $\mathfrak{l} := \mathfrak{g}$, $A := X_{\alpha,k}$ and $B := X_{\beta,h}$, taking Definition 3.1(f) into account. Indeed, in this case Definition 3.1(f) gives

$$C := [A, B] = [X_{\alpha,k}, X_{\beta,h}] = \sum_{t=1}^{\mu(\alpha+\beta)} c_{\alpha,k}^{\beta,h}(t) X_{\alpha+\beta,t}$$

Moreover, the assumptions $(2\alpha + \beta), (\alpha + 2\beta) \notin \Delta$ imply $[A, C] = 0 = [B, C]$. Thus we can apply Lemma 3.8(b) to expand $A^{(\ell)} B^{(m)} = X_{\alpha,k}^{(\ell)} X_{\beta,h}^{(m)}$; also, in the expansion we find we can still expand each divided power $C^{(q)} = \left(\sum_{t=1}^{\mu(\alpha+\beta)} c_{\alpha,k}^{\beta,h}(t) X_{\alpha+\beta,t} \right)^{(q)}$ via the formula in Lemma 3.8(d), which applies as $[c_{\alpha,k}^{\beta,h}(t') X_{\alpha+\beta,t'}, c_{\alpha,k}^{\beta,h}(t'') X_{\alpha+\beta,t''}] = 0$ for all t', t'' , by Proposition 2.21(a).

Relations (3.7) follow as an application of Lemma 3.8(c). Indeed, in the present case we can apply Lemma 3.8(c) to $\mathfrak{l} := \mathfrak{g}_0$, $L := X_{\alpha,1}$, $M := X_{\beta,h}$, so that

$$N := [L, M] = [X_{\alpha,1}, X_{\beta,h}] = \sum_{t=1}^{\mu(\alpha+\beta)} c_{\alpha,1}^{\beta,h}(t) X_{\alpha+\beta,t}, \quad T := 2^{-1} [L, N] = \epsilon X_{2\alpha+\beta,k}$$

Then in the formula of Lemma 3.8(c) we still have to expand $N^{(q)} = \left(\sum_{t=1}^{\mu(\alpha+\beta)} c_{\alpha,1}^{\beta,h}(t) X_{\alpha+\beta,t} \right)^{(q)}$ using Lemma 3.8(d), which again applies — all summands commute with each other — by the same arguments as above. In particular, $[M, N] = 0$ and $[L, T] = 0$ because of Proposition 2.21.

Finally, relations (3.8) — concerning roots of \mathfrak{g}_0 — are well-known by the classical theory. \square

(2) **Odd and even generators** (also involving the $X_{\tilde{\gamma}}$, $\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}$):

$$X_{\tilde{\gamma}} f(H) = f(H - \pi(\tilde{\gamma})(H)) X_{\tilde{\gamma}} \quad \forall \tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}, h \in \mathfrak{h}, f(T) \in \mathbb{K}[T] \quad (3.9)$$

$$X_{\gamma,k}^2 = 0 \quad \text{if all } c_{\gamma,k}^{\gamma,k}(t) \text{ are zero,} \quad X_{\gamma,k}^2 = 2^{-1} c_{\gamma,k}^{\gamma,k}(t') X_{2\gamma,t'} \quad \text{otherwise} \quad (3.10)$$

(with notation as in Definition 3.1(f.3.))

$$X_{\gamma,1} X_{-\gamma,k} = -X_{-\gamma,k} X_{\gamma,1} + H_{\sigma_{\gamma}(k)} \quad \forall \gamma \in \Delta_{-1} \quad (3.11)$$

with $H_{\sigma_{\gamma}(k)} = [X_{\gamma,1}, X_{-\gamma,k}] \in \mathfrak{h}_{\mathbb{Z}}$ as in Definition 3.1(d)

$$X_{\tilde{\gamma}} X_{\tilde{\eta}} = -X_{\tilde{\eta}} X_{\tilde{\gamma}} \quad \forall \tilde{\gamma}, \tilde{\eta} \in \tilde{\Delta}_{\bar{1}} : \pi(\tilde{\gamma}) + \pi(\tilde{\eta}) \notin \tilde{\Delta}, \forall \ell \in \mathbb{N} \quad (3.12)$$

$$X_{\gamma,k} X_{\eta,h} = -X_{\eta,h} X_{\gamma,k} + \sum_{t=1}^{\mu(\alpha+\gamma)} c_{\gamma,k}^{\eta,h}(t) X_{\gamma+\eta,t} \quad \forall \gamma, \eta \in \Delta_{\bar{1}} : \alpha + \gamma \in \Delta, \forall \ell \in \mathbb{N} \quad (3.13)$$

$$X_{\tilde{\alpha}}^{(\ell)} X_{\tilde{\gamma}} = X_{\tilde{\gamma}} X_{\tilde{\alpha}}^{(\ell)} \quad \forall \tilde{\alpha} \in \tilde{\Delta}_{\bar{0}}, \tilde{\gamma} \in \tilde{\Delta}_{\bar{1}} : \pi(\tilde{\alpha}) + \pi(\tilde{\gamma}) \notin \tilde{\Delta}, \forall \ell \in \mathbb{N} \quad (3.14)$$

$$X_{\alpha,k}^{(\ell)} X_{\gamma,h} = X_{\gamma,h} X_{\alpha,k}^{(\ell)} + \left(\sum_{t=1}^{\mu(\alpha+\gamma)} c_{\alpha,k}^{\gamma,h}(t) X_{\alpha+\gamma,t} \right) X_{\alpha,k}^{(\ell-1)} \quad (3.15)$$

$\forall \alpha \in \Delta_{\bar{0}}, \gamma \in \Delta_{\bar{1}} : \alpha + \gamma \in \Delta, 2\alpha + \gamma \notin \Delta, \forall \ell \in \mathbb{N}$

$$X_{\alpha,1}^{(\ell)} X_{\gamma,h} = X_{\gamma,h} X_{\alpha,1}^{(\ell)} + \left(\sum_{t=1}^{\mu(\alpha+\gamma)} c_{\alpha,1}^{\gamma,h}(t) X_{\alpha+\gamma,t} \right) X_{\alpha,1}^{(\ell-1)} + \epsilon X_{2\alpha+\gamma,t'} X_{\alpha,1}^{(\ell-2)} \quad (3.16)$$

$\forall \alpha \in \Delta_{\bar{0}}, \gamma \in \Delta_{\bar{1}} : \alpha + \gamma, 2\alpha + \gamma \in \Delta, \forall \ell, m \in \mathbb{N}$

where $c_{\gamma,k}^{\eta,h}(t)$, $c_{\alpha,k}^{\gamma,h}(t)$, $X_{\gamma+\eta,t}$, $X_{\alpha+\gamma,t}$, $\epsilon = \pm 1$ and the index t' (namely, the one such that $[X_{\alpha,1}, [X_{\alpha,1}, X_{\gamma,1}]] = \epsilon 2 X_{2\alpha+\gamma,t'}$) are given again as in Definition 3.1(f-g).

Proof. Almost all of these relations are proved much like those among even generators only.

A first exception is (3.10), which holds by Definition 3.1(h), taking into account that in the universal enveloping superalgebra one has $X^2 = X^{(2)}$ for every $X \in \mathfrak{g}_{\bar{1}}$. Another exception is (3.11), which is just another way of rewriting what is expressed in Definition 3.1(d).

As to the rest, relations (3.9), (3.12), (3.14) directly follow from definitions. Finally, relations (3.13), (3.15), (3.16) are proved, like relations (3.6) and (3.7), via induction like for Lemma 3.8. \square

Here now is our (super-version of) Kostant's theorem for $K_{\mathbb{Z}}(\mathfrak{g})$:

Theorem 3.9. *The Kostant superalgebra $K_{\mathbb{Z}}(\mathfrak{g})$ is a free \mathbb{Z} -module. More precisely, for any given total order \preceq of the set $\tilde{\Delta} \coprod \{1, \dots, n\}$ a \mathbb{Z} -basis of $K_{\mathbb{Z}}(\mathfrak{g})$ is the set \mathcal{B} of ordered “PBW-like monomials”, i.e. all products (without repetitions) of factors of type $X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})}$, $\binom{H_i}{n_i}$ and $X_{\tilde{\gamma}}$ — with $\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}}$, $i \in \{1, \dots, n\}$, $\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}$, and $\ell_{\tilde{\alpha}}, n_i \in \mathbb{N}$ — taken in the right order with respect to \preceq .*

Proof. Let us call “monomial” any product of several $X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})}$, several $\binom{H_i - z_i}{s_i}$ — with $z_i \in \mathbb{Z}$ — and several $X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}}$. For any such monomial, say \mathcal{M} , we consider the following three numbers:

- its “height” $ht(\mathcal{M})$, i.e. the sum of all $\ell_{\tilde{\alpha}}$ and $m_{\tilde{\gamma}}$ in \mathcal{M} ;
- its “factor number” $fac(\mathcal{M})$, defined to be the total number of factors (namely $X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})}$, $\binom{H_i - z_i}{n_i}$ or $X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}}$) within \mathcal{M} itself;
- its “inversion number” $inv(\mathcal{M})$, which is the sum of all inversions of the order \preceq among the indices of factors in \mathcal{M} when read from left to right.

We can now act upon any such \mathcal{M} with any of the following operations:

- (1) we move all factors $\binom{H_i - z_i}{s_i}$ to the leftmost position, by repeated use of (3.2) and (3.9): this produces a new monomial \mathcal{M}' , multiplied on the left by several (new) factors $\binom{H_i - \tilde{z}_i}{s_i}$;

– (2) whenever two *consecutive* factors $X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})}$ and $X_{\tilde{\alpha}'}^{(\ell_{\tilde{\alpha}'})}$ occur in \mathcal{M} , we replace their product in \mathcal{M} with an integral coefficient times a single factor, using (3.3); and similarly, we replace any pair of *consecutive* factors $X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}} X_{\tilde{\gamma}'}^{m_{\tilde{\gamma}'}}$ by the single factor $X_{\tilde{\gamma}+\tilde{\gamma}'}^{m_{\tilde{\gamma}}+m_{\tilde{\gamma}'}}$;

– (3) we replace any power $X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}}$ of an odd root vector with 0 or $\pm X_{2\gamma, t'}$, for $\tilde{\gamma} = (\gamma, k)$, whenever $m_{\gamma, k} > 1$, applying (3.10);

– (4) whenever two factors within \mathcal{M} occur side by side *in the wrong order w. r. to \preceq* , i.e. we have any one of the following situations

$$\begin{aligned} \mathcal{M} &= \cdots X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})} X_{\tilde{\beta}}^{(\ell_{\tilde{\beta}})} \cdots, & \mathcal{M} &= \cdots X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})} X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}} \cdots \\ \mathcal{M} &= \cdots X_{\tilde{\eta}}^{m_{\tilde{\eta}}} X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})} \cdots, & \mathcal{M} &= \cdots X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}} X_{\tilde{\eta}}^{m_{\tilde{\eta}}} \cdots \end{aligned}$$

with $\tilde{\alpha} \succneq \tilde{\beta}$, $\tilde{\alpha} \succneq \tilde{\gamma}$, $\tilde{\eta} \succneq \tilde{\alpha}$ and $\tilde{\gamma} \succneq \tilde{\eta}$ respectively, we can use all relations (3.4–8) and (3.11–16) to re-write this product of two distinguished factors, so that all of \mathcal{M} expands into a \mathbb{Z} –linear combination of new monomials. In some cases one has to read these relations the other way round: for instance, one can use (3.7) to re-write $X_{\beta, h}^{(m)} X_{\alpha, 1}^{(\ell)}$ as $X_{\beta, h}^{(m)} X_{\alpha, 1}^{(\ell)} = \cdots$.

By definition, $K_{\mathbb{Z}}(\mathfrak{g})$ is \mathbb{Z} –spanned by all (unordered) monomials in the $X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})}$, the $\binom{H_i}{n_i}$ and the $X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}}$. Let \mathcal{M} be any one of these monomials: it is PBW-like, i.e. in \mathcal{B} , if and only if no one of steps (2) to (4) may be applied; but if not, we now see what is the effect of applying such steps.

Applying step (1) gives $\mathcal{M} = \mathcal{H} \mathcal{M}'$ where \mathcal{H} is some product of $\binom{H_i - z_i}{s_i}$, and \mathcal{M}' is a new monomial such that $ht(\mathcal{M}') = ht(\mathcal{M})$, $fac(\mathcal{M}') \leq fac(\mathcal{M})$, $inv(\mathcal{M}') \leq inv(\mathcal{M})$, and the strict inequality in the middle holds if and only if $\mathcal{H} \neq 1$, i.e. step (1) is non-trivial. Actually, this is clear at once when one realizes that \mathcal{M}' is nothing but “ \mathcal{M} with all factors $\binom{H_i - z_i}{s_i}$ removed.”

Then we apply any one of steps (2), (3) or (4) to \mathcal{M}' .

Step (2), if non-trivial, yields $\mathcal{M}' = z \mathcal{M}^{\vee}$, for some $z \in \mathbb{Z}$ and some monomial \mathcal{M}^{\vee} such that $ht(\mathcal{M}^{\vee}) = ht(\mathcal{M}')$, $fac(\mathcal{M}^{\vee}) \leq fac(\mathcal{M}')$. Instead, step (3), still if non-trivial, gives $\mathcal{M}' = 0$.

Finally, step (4) gives $\mathcal{M}' = \mathcal{M}^{\vee} + \sum_k z_k \mathcal{M}_k$, where $z_k \in \mathbb{Z}$ (for all k) and \mathcal{M}^{\vee} and the \mathcal{M}_k are monomials such that $ht(\mathcal{M}_k) \leq ht(\mathcal{M}^{\vee}) \forall k$, $ht(\mathcal{M}^{\vee}) = ht(\mathcal{M}')$, $inv(\mathcal{M}^{\vee}) \leq inv(\mathcal{M}')$.

In short, through either step (2), or (3), or (4), we achieve an expansion

$$\mathcal{M}' = \sum_h z'_h \mathcal{H}_h \mathcal{M}'_h, \quad z'_h \in \mathbb{Z} \quad \forall h \quad (3.17)$$

(the sum in right-hand side possibly being void, hence equal to zero) where — unless the step is trivial, for then we get all equalities — we have

$$\left(ht(\mathcal{M}'_h) \leq ht(\mathcal{M}') \right) \vee \left(fac(\mathcal{M}'_h) \leq fac(\mathcal{M}') \right) \vee \left(inv(\mathcal{M}'_h) \leq inv(\mathcal{M}') \right) \quad (3.18)$$

Now we repeat, applying step (1) and then step (2) or (3) or (4) to every monomial \mathcal{M}'_h in (3.17). Then we iterate, until we get only monomials whose inversion number is zero: by (3.18), this is possible indeed, and it is achieved after finitely many iterations. The outcome reads

$$\mathcal{M}' = \sum_j z_j \mathcal{H}_j'' \mathcal{M}_j'', \quad z_j \in \mathbb{Z} \quad \forall j \quad (3.19)$$

where $inv(\mathcal{M}_j'') = 0$ for every index j , i.e. all monomials \mathcal{M}_j'' are ordered and without repetitions, that is they belong to \mathcal{B} . Now each \mathcal{H}_j'' belongs to $\mathbb{H}_{\mathbb{Z}}$ (notation of Corollary 3.5), just by construction. Then Corollary 3.5 ensures that each \mathcal{H}_j'' expand into a \mathbb{Z} –linear combination of ordered monomials in the $\binom{H_i}{n_i}$. Therefore (3.19) yields

$$\mathcal{M} = \sum_s \hat{z}_s \mathcal{H}_s^{\wedge} \mathcal{M}_s^{\wedge}, \quad \hat{z}_s \in \mathbb{Z} \quad \forall s \quad (3.20)$$

where every \mathcal{H}_s^{\wedge} is an ordered monomial, without repetitions, in the $\binom{H_i}{n_i}$, while for each index s we have $\mathcal{M}_s^{\wedge} = \mathcal{M}_j''$ for some j — those in (3.19).

Using again relations (3.2) and (3.9), we can switch positions among the factors $\binom{H_i}{n_i}$ in \mathcal{H}_s^{\wedge} and the factors in \mathcal{M}_s^{\wedge} (for each s), so to get a new monomial \mathcal{M}_s° which is ordered, without

repetitions, but might have factors of type $\binom{H_i - z_i}{m_i}$ with $z_i \in \mathbb{Z} \setminus \{0\}$ — so that $\mathcal{M}_s^c \notin \mathcal{B}$. But then $\binom{H_i - z_i}{m_i} \in \mathbb{H}_{\mathbb{Z}}$, hence again by Corollary 3.5 that factor expands into a \mathbb{Z} -linear combination of ordered monomials, without repetitions, in the $\binom{H_i}{\ell}$. Plugging every such expansion inside each monomial \mathcal{M}_s^\wedge instead of each factor $\binom{H_i - z_i}{m_i}$ — $i = 1, \dots, r$ — we eventually find

$$\mathcal{M} = \sum_q z_q \mathcal{M}_q^! , \quad z_q \in \mathbb{Z} \quad \forall q$$

where now every $\mathcal{M}_q^!$ is a PBW-like monomial, i.e. $\mathcal{M}_q^! \in \mathcal{B}$ for every q .

As $K_{\mathbb{Z}}(\mathfrak{g})$, by definition, is spanned over \mathbb{Z} by all monomials in the $X_{\tilde{\alpha}}^{(\ell_{\tilde{\alpha}})}$, the $\binom{H_i}{n_i}$ and the $X_{\tilde{\gamma}}^{m_{\tilde{\gamma}}}$, our analysis yields $K_{\mathbb{Z}}(\mathfrak{g}) \subseteq \text{Span}_{\mathbb{Z}}(\mathcal{B})$. On the other hand, by definition $\text{Span}_{\mathbb{Z}}(\mathcal{B})$ in turn is contained in $K_{\mathbb{Z}}(\mathfrak{g})$. Therefore $K_{\mathbb{Z}}(\mathfrak{g}) = \text{Span}_{\mathbb{Z}}(\mathcal{B})$, i.e. \mathcal{B} spans $K_{\mathbb{Z}}(\mathfrak{g})$ over \mathbb{Z} .

At last, the PBW theorem for Lie superalgebras over fields ensures that \mathcal{B} is a \mathbb{K} -basis for $U(\mathfrak{g})$, as $B := \{H_1, \dots, H_r\} \amalg \{X_{\tilde{\alpha}} \mid \tilde{\alpha} \in \tilde{\Delta}\}$ is a \mathbb{K} -basis of \mathfrak{g} (cf. [18]). So \mathcal{B} is linearly independent over \mathbb{K} , hence over \mathbb{Z} . Therefore \mathcal{B} is a \mathbb{Z} -basis for $K_{\mathbb{Z}}(\mathfrak{g})$, and the latter is a free \mathbb{Z} -module. \square

3.10. Kostant superalgebras for special sub-objects. We can consider \mathbb{Z} -integral forms for the sub-objects, as follows. First fix a Chevalley basis B and $\mathfrak{g}^{\mathbb{Z}}$ as in Definition 3.1. Second, for $\alpha \in \Delta$, $q \geq -1$, we set $B_{\alpha} := B \cap \mathfrak{g}_{\alpha}$, $\mathfrak{g}_{\alpha}^{\mathbb{Z}} := \text{Span}_{\mathbb{Z}}(B_{\alpha})$, $\mathfrak{g}_q^{\mathbb{Z}} := \text{Span}_{\mathbb{Z}}(\amalg_{ht(\alpha)=q} B_{\alpha})$, and

$$\begin{aligned} \mathfrak{g}_0^{\mathbb{Z}} &:= \mathfrak{h}_{\mathbb{Z}} \oplus \left(\bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_{\alpha}^{\mathbb{Z}} \right), & \mathfrak{g}_1^{\mathbb{Z}} &:= \bigoplus_{\gamma \in \Delta_1} \mathfrak{g}_{\gamma}^{\mathbb{Z}}, & \mathfrak{g}_0^{\mathbb{Z}} &:= \mathfrak{h}_{\mathbb{Z}} \oplus \left(\bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_{\alpha}^{\mathbb{Z}} \right) \\ \mathfrak{g}_{0\uparrow}^{\mathbb{Z}} &:= \bigoplus_{\alpha \in \Delta_{0\uparrow}} \mathfrak{g}_{\alpha}^{\mathbb{Z}}, & \mathfrak{g}_{1\uparrow}^{\mathbb{Z}} &:= \bigoplus_{\gamma \in \Delta_{1\uparrow}} \mathfrak{g}_{\gamma}^{\mathbb{Z}}, & \mathfrak{g}_{t\uparrow}^{\mathbb{Z}} &:= \bigoplus_{q>t} \mathfrak{g}_q^{\mathbb{Z}} \quad (t \geq -1), & \mathfrak{g}_{-1,0}^{\mathbb{Z}} &:= \mathfrak{g}_{-1}^{\mathbb{Z}} \oplus \mathfrak{g}_0^{\mathbb{Z}} \end{aligned}$$

(the last only for $\mathfrak{g} \not\cong \tilde{S}(n)$; notation of §2.22) with $\mathfrak{h}_{\mathbb{Z}} := \text{Span}_{\mathbb{Z}}(H_1, \dots, H_r)$, cf. Definition 3.1(a).

Definition 3.11. We define $K_{\mathbb{Z}}(\mathfrak{g}_{-1\uparrow})$ as the unital \mathbb{Z} -subsuperalgebra of $U(\mathfrak{g})$ generated by $\binom{H_i}{m}$, $X_{\tilde{\alpha}}^{(m)}$ and $X_{\tilde{\gamma}}$ for all $m \in \mathbb{N}$, $1 \leq i \leq r$, $\tilde{\alpha} \in \tilde{\Delta}_0$, $\tilde{\gamma} \in \tilde{\Delta}_{1\uparrow}$. In a similar way, we define the unital \mathbb{Z} -subsuperalgebras $K_{\mathbb{Z}}(\mathfrak{g}_0)$, $K_{\mathbb{Z}}(\mathfrak{g}_0\uparrow)$, $K_{\mathbb{Z}}(\mathfrak{g}_{t\uparrow})$ — $t \geq 0$ — and $K_{\mathbb{Z}}(\mathfrak{g}_{-1,0})$ — for $\mathfrak{g} \not\cong \tilde{S}(n)$ — as the ones generated by the binomial coefficients, divided powers of even root vectors, and odd root vectors involved in the very definition of \mathfrak{g}_0 , $\mathfrak{g}_0\uparrow$, $\mathfrak{g}_{t\uparrow}$ and $\mathfrak{g}_{-1,0}$.

Also, we denote $\bigwedge \mathfrak{g}_1^{\mathbb{Z}}$, resp. by $\bigwedge \mathfrak{g}_{1\uparrow}^{\mathbb{Z}}$, the (unital) exterior \mathbb{Z} -algebra over $\mathfrak{g}_1^{\mathbb{Z}}$, resp. over $\mathfrak{g}_{1\uparrow}^{\mathbb{Z}}$.

All these objects are related by the following consequence of Theorem 3.9 (in particular, the first isomorphism is an integral version of the factorization $U(\mathfrak{g}) \cong U(\mathfrak{g}_0) \otimes_{\mathbb{K}} \bigwedge \mathfrak{g}_1$, see [18]), whose proof follows from the arguments used for Theorem 3.9, or as a direct consequence of it:

Corollary 3.12. *There exist isomorphisms of \mathbb{Z} -modules*

$$\begin{aligned} K_{\mathbb{Z}}(\mathfrak{g}) &\cong K_{\mathbb{Z}}(\mathfrak{g}_0) \otimes_{\mathbb{Z}} \bigwedge \mathfrak{g}_1^{\mathbb{Z}}, & K_{\mathbb{Z}}(\mathfrak{g}_{-1\uparrow}) &\cong K_{\mathbb{Z}}(\mathfrak{g}_0) \otimes_{\mathbb{Z}} \bigwedge \mathfrak{g}_{1\uparrow}^{\mathbb{Z}}, & K_{\mathbb{Z}}(\mathfrak{g}_{0\uparrow}) &\cong K_{\mathbb{Z}}(\mathfrak{g}_{0\uparrow}) \otimes_{\mathbb{Z}} \bigwedge \mathfrak{g}_{1\uparrow}^{\mathbb{Z}} \\ \text{and of } \mathbb{Z}\text{-superalgebras } & K_{\mathbb{Z}}(\mathfrak{g}_0) &\cong K_{\mathbb{Z}}(\mathfrak{g}_0) \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g}_{0\uparrow}), & K_{\mathbb{Z}}(\mathfrak{g}_{-1,0}) &\cong K_{\mathbb{Z}}(\mathfrak{g}_0) \otimes_{\mathbb{Z}} \bigwedge \mathfrak{g}_{-1}^{\mathbb{Z}}. \end{aligned}$$

Remark 3.13. Following a classical pattern, one defines the *superalgebra of distributions* $\mathcal{D}ist(G)$ on any supergroup G , by an obvious extension of the standard notion in the even setting; see [2], §4, for details. If G is any one of the algebraic supergroups (over \mathbb{k}) that we are going to construct, then $\text{Lie}(G) = \mathfrak{g}$ — with some more precisions: see subsection 4.6 later on. Then one can check — like in [2], §4 — that $\mathcal{D}ist(G) = \mathbb{k} \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g}) =: K_{\mathbb{k}}(\mathfrak{g})$. An entirely similar remark occurs when the supergroup G is one of the “Chevalley supergroups” introduced in [9].

Any morphism $\varphi : G' \rightarrow G''$ between two supergroups induces (functorially) a morphism $D_{\varphi} : \mathcal{D}ist(G') \rightarrow \mathcal{D}ist(G'')$, which is injective whenever φ is injective. If in addition the supergroups G' and G'' are of the type mentioned above, then $\mathcal{D}ist(G') = K_{\mathbb{k}}(\mathfrak{g}')$ and $\mathcal{D}ist(G'') = K_{\mathbb{k}}(\mathfrak{g}'')$, so that $D_{\varphi} : K_{\mathbb{k}}(\mathfrak{g}') \rightarrow K_{\mathbb{k}}(\mathfrak{g}'')$, which is an embedding if G' is a subsupergroup of G'' .

3.4 Admissible lattices for \mathfrak{g} -modules

The tools of our construction of algebraic supergroups are the Lie superalgebra \mathfrak{g} together with an integrable \mathfrak{g} -module. As we need an integral version of \mathfrak{g} — and even more, of $U(\mathfrak{g})$, namely the Kostant superalgebra — we also need a suitable integral form of any integrable \mathfrak{g} -module.

Let a Chevalley basis B and Kostant algebra $K_{\mathbb{Z}}(\mathfrak{g})$ be given, as before. If V a \mathbb{K} -vector space we call a subset $M \subseteq V$ a \mathbb{Z} -form of V if (as usual) $M = \text{Span}_{\mathbb{Z}}(\mathcal{B})$ for some \mathbb{K} -basis \mathcal{B} of V .

Definition 3.14. Let V be a \mathfrak{g} -module. We retain terminology and notation of Definition 2.25.

(a) We call V *rational* if $\mathfrak{h}_{\mathbb{Z}} := \text{Span}_{\mathbb{Z}}(H_1, \dots, H_r)$ acts diagonally on V with eigenvalues in \mathbb{Z} ; in other words, $V = \bigoplus_{\mu \in \mathfrak{h}^*} V_{\mu}$ is a weight \mathfrak{g} -module and $\mu(H_i) \in \mathbb{Z}$ for all i and all $\mu \in \text{Supp}(V)$.

(b) Any \mathbb{Z} -lattice M of V is said to be *admissible* if it is a $K_{\mathbb{Z}}(\mathfrak{g})$ -stable \mathbb{Z} -form of V .

Note that — by the last remark in Definition 2.25 — the \mathfrak{g} -action on any rational \mathfrak{g} -module V can be extended to a $\bar{\mathfrak{g}}$ -action so that \mathcal{E} acts diagonally (=semisimply) on V with eigenvalues in \mathbb{Z} ; in short, V itself is also a *rational $\bar{\mathfrak{g}}$ -module*, with $\bar{\mathfrak{h}}$ -weight space decomposition $V = \bigoplus_{\nu \in \bar{\mathfrak{h}}^*} V_{\nu}$.

The first property of admissible lattices is natural (its proof being classical, cf. [17], §2, Cor. 1):

Proposition 3.15. *Let V be a weight \mathfrak{g} -module. Then any admissible lattice M of V is the direct sum of its \mathfrak{h} -weight components, i.e. $M = \bigoplus_{\lambda \in \mathfrak{h}^*} (M \cap V_{\lambda})$, and similarly for $\bar{\mathfrak{h}}$ -weights.*

Next property instead is an existence result, under mild conditions:

Proposition 3.16. *Let V be a finite dimensional, completely reducible \mathfrak{g} -module. Then V is a weight module. If it is also rational, then there exists an admissible lattice M of it.*

Proof. First of all, by Remark 2.26 V is a weight module. Now assume it is also rational. Then by the complete reducibility assumption we can reduce to assume V irreducible. In that case, like for [16], Theorem 3.1, we find that V is *cyclic*, i.e. it can be generated by a single vector, and the latter can be taken to be a highest weight vector (cf. Definition 2.25). Letting v be such a highest weight vector, set $M := K_{\mathbb{Z}}(\mathfrak{g}) \cdot v$: then one can repeat the classical proof — like in [17], §2, Corollary 1 — and eventually show that such an M is indeed an admissible lattice of V as required. \square

We can also describe the stabilizer of an admissible lattice:

Proposition 3.17. *Let V be a faithful, rational, finite dimensional \mathfrak{g} -module, M an admissible lattice of V , and $\mathfrak{g}_V = \{X \in \mathfrak{g} \mid X \cdot M \subseteq M\}$ its stabilizer. Then, letting $\mathfrak{h}_V := \{H \in \mathfrak{h} \mid \mu(H) \in \mathbb{Z}, \forall \mu \in \Lambda\}$, where Λ is the set of weights of V , we have $\mathfrak{g}_V = \mathfrak{h}_V \oplus \left(\bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} \mathbb{Z} X_{\tilde{\alpha}} \right)$.*

In particular, \mathfrak{g}_V is a lattice in \mathfrak{g} , independent of the choice of the admissible lattice M .

Proof. The classical proof in [17], §2, Corollary 2, applies again, with some additional arguments to manage odd root spaces. Indeed, the same arguments as in [loc. cit.] prove that $\mathfrak{g}_V = \mathfrak{h}_V \oplus \left(\bigoplus_{\tilde{\alpha} \in \tilde{\Delta}} (\mathfrak{g}_V \cap \mathbb{K} X_{\tilde{\alpha}}) \right)$; then one still has to prove that $\mathfrak{g}_V \cap \mathbb{K} X_{\tilde{\alpha}} = \mathbb{Z} X_{\tilde{\alpha}}$ for all $\tilde{\alpha} \in \tilde{\Delta}$. The arguments in [loc. cit.] show that $\mathfrak{g}_V \cap \mathbb{K} X_{\tilde{\alpha}}$ is a cyclic \mathbb{Z} -submodule of \mathfrak{g}_V which may be \mathbb{Z} -spanned by some $\frac{1}{n_{\tilde{\alpha}}} X_{\tilde{\alpha}}$ with $n_{\tilde{\alpha}} \in \mathbb{N}_+$ (for $\tilde{\alpha} \in \tilde{\Delta}$). What is left to prove is that $n_{\tilde{\alpha}} = 1$.

For every $\tilde{\alpha} \in \tilde{\Delta}_0$ the same arguments as in [loc. cit.] still yield $n_{\tilde{\alpha}} = 1$.

For every $\tilde{\alpha} \in \tilde{\Delta}_{-1}$, by Definition 3.1(f.2) we can find $\tilde{\beta} \in \tilde{\Delta}$ such that $\alpha + \beta \in \Delta_0$, with $\alpha := \pi(\tilde{\alpha})$, $\beta := \pi(\tilde{\beta})$, and $[X_{\tilde{\alpha}}, X_{\tilde{\beta}}] = \pm X_{(\alpha+\beta, t')}$ for some $t' \in \{1, \dots, \mu(\alpha+\beta)\}$. This yields $\pm \frac{1}{n_{\tilde{\alpha}}} X_{(\alpha+\beta, t')} = \left[\frac{1}{n_{\tilde{\alpha}}} X_{\tilde{\alpha}}, X_{\tilde{\beta}} \right] \in [\mathfrak{g}_V, \mathfrak{g}^{\mathbb{Z}}] \subseteq [\mathfrak{g}_V, \mathfrak{g}_V] \subseteq \mathfrak{g}_V$ because $\mathfrak{g}^{\mathbb{Z}} \subseteq \mathfrak{g}_V$ and \mathfrak{g}_V is a Lie subsuperalgebra of \mathfrak{g} . Therefore $\frac{1}{n_{\tilde{\alpha}}} X_{(\alpha+\beta, t')} \in \mathfrak{g}_V \cap \mathbb{K} X_{(\alpha+\beta, t')} = \mathbb{Z} X_{(\alpha+\beta, t')}$ because $X_{(\alpha+\beta, t')} \in \tilde{\Delta}_0$ — and thanks to the previous step — which eventually forces $n_{\tilde{\alpha}} = 1$.

Finally, consider $\tilde{\alpha} \in \tilde{\Delta}_z$ with $z \geq 1$, and let $\alpha := \pi(\tilde{\alpha}) \in \Delta_z$. Then, by direct analysis, we see that there exists $\gamma \in \Delta_{-1}$ such that $(\alpha + \gamma) \in \Delta_{z-1}$; therefore, for $\tilde{\gamma} := (\gamma, 1)$ we have again by

Definition 3.1(f.2) that $[X_{\tilde{\alpha}}, X_{\tilde{\gamma}}] = \pm X_{(\alpha+\gamma, t')}$. Just like before, this yields $\pm \frac{1}{n_{\tilde{\alpha}}} X_{(\alpha+\gamma, t')} = \left[\frac{1}{n_{\tilde{\alpha}}} X_{\tilde{\alpha}}, X_{\tilde{\gamma}} \right] \in [\mathfrak{g}_V, \mathfrak{g}^{\mathbb{Z}}] \subseteq [\mathfrak{g}_V, \mathfrak{g}_V] \subseteq \mathfrak{g}_V$ hence $\frac{1}{n_{\tilde{\alpha}}} X_{(\alpha+\gamma, t')} \in \mathfrak{g}_V \cap \mathbb{K} X_{(\alpha+\gamma, t')}$. Since α has height $|\alpha| = z$ and $|\alpha+\gamma| = z-1$, by induction on z we assume $\mathfrak{g}_V \cap \mathbb{K} X_{(\alpha+\gamma, t')} = \mathbb{Z} X_{(\alpha+\gamma, t')}$: the basis of induction is $z = 0$ which corresponds to roots in Δ_0 , that we already disposed of. Therefore $\frac{1}{n_{\tilde{\alpha}}} X_{(\alpha+\gamma, t')} \in \mathbb{Z} X_{(\alpha+\gamma, t')}$, so that $n_{\tilde{\alpha}} = 1$. \square

4 Algebraic supergroups G_V of Cartan type

Classically, Chevalley groups are defined as follows. Fix a finite dimensional semisimple Lie algebra \mathfrak{g} over an algebraically closed field \mathbb{K} of characteristic zero, a Chevalley basis of \mathfrak{g} and the associated Kostant form $K_{\mathbb{Z}}(\mathfrak{g})$ of $U(\mathfrak{g})$. Then any simple finite dimensional \mathfrak{g} -module V contains a \mathbb{Z} -lattice M , which is $K_{\mathbb{Z}}(\mathfrak{g})$ -stable, so $K_{\mathbb{Z}}(\mathfrak{g})$ acts on M . Using this action and its extensions by scalars to any field \mathbb{k} , one defines one-parameter subgroups $x_{\alpha}(t)$, for all roots α and $t \in \mathbb{k}$, within $\mathrm{GL}(V_{\mathbb{k}})$, $V_{\mathbb{k}} := M \otimes \mathbb{k}$: the Chevalley group (associated with \mathfrak{g} and V) is the subgroup of $\mathrm{GL}(V_{\mathbb{k}})$ generated by the $x_{\alpha}(t)$. If one has to extend this construction to that of a genuine \mathbb{Z} -group scheme, then slight variations are in order, e.g. one has to “add by hand” a maximal torus.

This construction has been adapted to simple Lie superalgebras \mathfrak{g} of classical type in [9], [11]. We do now the same for all simple Lie superalgebras \mathfrak{g} of Cartan type.

4.1 One-parameter supersubgroups

The supergroups we look for will be realized as subgroup-functors of some linear supergroup functors $\mathrm{GL}(V)$, generated by suitable subgroup functors: these are super-analogues of one-parameter subgroups in the classical theory, thus we call them “one-parameter supersubgroups”. Like in the classical setup, they will be of two types: multiplicative and additive: the latter ones then will split into two more types, according to the type (even or odd) of the roots involved.

We retain the notation of sections 2 and 3. In particular, V is a fixed faithful, rational, finite dimensional weight \mathfrak{g} -module with an admissible lattice M in it (e.g., if V is completely reducible).

Fix a commutative unital \mathbb{Z} -algebra \mathbb{k} , and set $\mathfrak{g}_{V, \mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}} \mathfrak{g}_V$, $V_{\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}} M$, $U_{\mathbb{k}}(\mathfrak{g}) := \mathbb{k} \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g})$. Then $\mathfrak{g}_{V, \mathbb{k}}$ acts faithfully on $V_{\mathbb{k}}$, which yields a Lie superalgebra monomorphism $\mathfrak{g}_{V, \mathbb{k}} \hookrightarrow \mathrm{End}(V_{\mathbb{k}})$ and a superalgebra morphism $U_{\mathbb{k}}(\mathfrak{g}) \rightarrow \mathrm{End}(V_{\mathbb{k}})$. Now, for every $A \in (\mathrm{salg})_{\mathbb{k}}$ define $\mathfrak{g}_A := A \otimes_{\mathbb{k}} \mathfrak{g}_{V, \mathbb{k}}$ ($= A \otimes_{\mathbb{Z}} \mathfrak{g}_V$), $V_A := A \otimes_{\mathbb{k}} V_{\mathbb{k}}$ ($= A \otimes_{\mathbb{Z}} M$) and $U_A(\mathfrak{g}) := A \otimes_{\mathbb{k}} U_{\mathbb{k}}(\mathfrak{g})$ ($= A \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{g})$). Then \mathfrak{g}_A acts faithfully on V_A , which yields morphisms $\mathfrak{g}_A \hookrightarrow \mathrm{End}(V_A)$ and $U_A(\mathfrak{g}) \rightarrow \mathrm{End}(V_A)$. Moreover (as sketched in §2.9) all these constructions are functorial in A .

The splitting $V = \oplus_{\mu \in \mathfrak{h}^*} V_{\mu}$ of the \mathfrak{g} -module V into \mathfrak{h} -weight spaces yields, for any $A \in (\mathrm{salg})_{\mathbb{k}}$, a similar splitting $V_{\mathbb{k}}(A) = \oplus_{\mu \in \mathfrak{h}^*} V_{\mu}(A)$ — using notation as in Examples 2.6(a). Now fix any element $H \in \mathfrak{h}_{\mathbb{Z}} := \mathrm{Span}_{\mathbb{Z}}(H_1, \dots, H_r)$ — see Definition 3.1(a): then $\mu(H) \in \mathbb{Z}$ for any $\mu \in \mathrm{Supp}(V)$, as V is rational. Let $U(A_{\bar{0}})$ the group of invertible elements in $A_{\bar{0}}$: we set

$$h_H(u).v := u^{\mu(H)} v \quad \forall v \in V_{\mu}(A), \mu \in \mathrm{Supp}(V), u \in U(A_{\bar{0}})$$

which defines an operator $u^H := h_H(u) \in \mathrm{GL}(V_{\mathbb{k}}(A))$ for all $u \in U(A_{\bar{0}})$.

Note that the formal identity $u^H = \sum_{m=0}^{+\infty} (u-1)^m \binom{H}{m}$, whose right-hand side becomes a finite sum if acting on a single weight space $V_{\mu}(A)$, shows that the operator $u^H = h_H(u)$ is one of those given by the $U_A(\mathfrak{g})$ -action on V . Note also that $H = \sum_{i=1}^r z_i H_i$ ($z_i \in \mathbb{Z}$) yields $h_H(u) := \prod_{i=1}^r h_{H_i}(u)^{z_i}$.

Definition 4.1. For any $H \in \mathfrak{h}_{\mathbb{Z}}$, we define the supergroup functor h_H — also referred to as a “multiplicative one-parameter supersubgroup” — from $(\mathrm{salg})_{\mathbb{k}}$ to (groups) as given on objects by $h_H(A) := \{u^H := h_H(u) \mid u \in U(A_{\bar{0}})\}$ and given on morphisms in the obvious way.

We also write $h_i := h_{H_i}$ for $i = 1, \dots, r$, and $h_{\alpha} := h_{H_{\alpha}}$ for $\alpha \in \Delta$.

Let $\tilde{\alpha} \in \tilde{\Delta}_0$, $\tilde{\beta} \in \tilde{\Delta}_1$, and let $X_{\tilde{\alpha}}$ and $X_{\tilde{\beta}}$ be associated root vectors in B . Both $X_{\tilde{\alpha}}$ and $X_{\tilde{\beta}}$ act as nilpotent operators on V , thus on M and $V_{\mathbb{k}}$; the same holds for

$$tX_{\tilde{\alpha}}, \vartheta X_{\tilde{\beta}} \in \text{End}(V_{\mathbb{k}}(A)) \quad \forall t \in A_0, \vartheta \in A_1. \quad (4.1)$$

Since $(\vartheta X_{\tilde{\beta}})^2 = -\vartheta^2 X_{\tilde{\beta}}^2 = 0$, we have $Y^m/m! \in (K_{\mathbb{Z}}(\mathfrak{g}))(A)$ for any $m \in \mathbb{N}$, $Y \in \{tX_{\tilde{\alpha}}, \vartheta X_{\tilde{\beta}}\}$ as in (4.1); moreover, $Y^m/m!$ acts as zero for $m \gg 0$, by nilpotency, so $\exp(Y) := \sum_{m=0}^{+\infty} Y^m/m!$ in $\text{GL}(V_{\mathbb{k}}(A))$ is well defined. In particular, $\exp(\vartheta X_{\tilde{\beta}}) := \sum_{m=0}^{+\infty} (\vartheta X_{\tilde{\beta}})^m/m! = 1 + \vartheta X_{\tilde{\beta}}$.

Definition 4.2. Let $\tilde{\alpha} \in \tilde{\Delta}_0$, $\tilde{\beta} \in \tilde{\Delta}_1$, and let $X_{\tilde{\alpha}}$ and $X_{\tilde{\beta}}$ be as above; then set $x_{\tilde{\alpha}}(t) := \exp(tX_{\tilde{\alpha}}) = 1 + tX_{\tilde{\alpha}} + t^2X_{\tilde{\alpha}}^{(2)} + \dots$, for all $t \in A_0$, and $x_{\tilde{\beta}}(\vartheta) := \exp(\vartheta X_{\tilde{\beta}}) = 1 + \vartheta X_{\tilde{\beta}}$, for all $\vartheta \in A_1$. We define the supergroup functors $x_{\tilde{\alpha}}$ and $x_{\tilde{\beta}}$ from $(\text{salg})_{\mathbb{k}}$ to (groups) setting them on objects as $x_{\tilde{\alpha}}(A) := \{x_{\tilde{\alpha}}(t) \mid t \in A_0\}$, $x_{\tilde{\beta}}(A) := \{x_{\tilde{\beta}}(\vartheta) \mid \vartheta \in A_1\}$ for all $A \in (\text{salg})_{\mathbb{k}}$ — the definition on morphisms then should be clear.

In order to unify the notation, we shall denote by $x_{\tilde{\eta}}(\mathbf{t})$, for $\tilde{\eta} \in \tilde{\Delta}$, any one of the two possibilities above, so that $\mathbf{t} \in A_0 \cup A_1$. Finally, for later convenience we shall also formally write $x_{\tilde{\zeta}}(\mathbf{t}) := 1$ when $\pi(\tilde{\zeta})$ belongs to the \mathbb{Z} -span of Δ but $\pi(\tilde{\zeta}) \notin \Delta$.

As in the Lie supergroup setting (see subsection 2.3 in [9]), one can easily prove the following:

Proposition 4.3. *The following hold:*

(a) *Every supergroup functor h_H is representable, so it is an affine supergroup, of (super)-dimension $1|0$. Indeed, $h_H(A) = \text{Hom}(\mathbb{k}[z, z^{-1}], A)$, for $A \in (\text{salg})_{\mathbb{k}}$, with $\Delta(z^{\pm 1}) = z^{\pm 1} \otimes z^{\pm 1}$.*

(b) *The supergroup functors $x_{\tilde{\alpha}}$ and $x_{\tilde{\beta}}$ are representable, so they are affine supergroups, respectively of (super)dimension $1|0$ and $0|1$. Indeed, for every $A \in (\text{salg})_{\mathbb{k}}$ one has $x_{\tilde{\alpha}}(A) = \text{Hom}(\mathbb{k}[x], A)$ with $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $x_{\tilde{\beta}}(A) = \text{Hom}(\mathbb{k}[\xi], A)$ with $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$.*

(Remark: in both cases, Δ denotes the comultiplication in the Hopf superalgebra under exam)

4.2 Construction of supergroups G_V of Cartan type

We now define our supergroups of Cartan type as suitable subgroup functors — from $(\text{salg})_{\mathbb{k}}$ to (groups) — of $\text{GL}(V_{\mathbb{k}})$. Further details about the formalism of (Grothendieck) topologies in categories and sheafification of functors can be found in [9], Appendix, and in references therein.

Once and for all, we let \mathfrak{g} and V as above, and we fix also a partition $\Delta = \Delta^+ \amalg \Delta^-$ of the roots into positive and negative ones as in §2.22.

Definition 4.4. We call *Cartan (type) supergroup functor*, associated with \mathfrak{g} and V , the functor $G_V : (\text{salg})_{\mathbb{k}} \rightarrow (\text{groups})$ defined as follows. Let $A, B \in \text{Ob}((\text{salg})_{\mathbb{k}})$, $\phi \in \text{Hom}_{(\text{salg})_{\mathbb{k}}}(A, B)$: then

— the object $G_V(A)$ is the subgroup of $\text{GL}(V_{\mathbb{k}}(A))$ generated by the subgroups $h_H(A)$ and $x_{\tilde{\alpha}}(A)$, with $H \in \mathfrak{h}_{\mathbb{Z}}$, $\tilde{\alpha} \in \tilde{\Delta}$, i.e. $G_V(A) := \left\langle h_H(A), x_{\tilde{\alpha}}(A) \right\rangle_{H \in \mathfrak{h}_{\mathbb{Z}}, \tilde{\alpha} \in \tilde{\Delta}} = \left\langle h_i(A), x_{\tilde{\alpha}}(A) \right\rangle_{\substack{\tilde{\alpha} \in \tilde{\Delta} \\ i=1, \dots, r;}}$;

— the morphism $G_V(A) \xrightarrow{G_V(\phi)} G_V(B)$ is the restriction of $\text{GL}(V_{\mathbb{k}}(A)) \rightarrow \text{GL}(V_{\mathbb{k}}(B))$, the morphism induced by ϕ by functoriality of $\text{GL}(V_{\mathbb{k}})$ (which maps the generators of $G_V(A)$ to those of $G_V(B)$).

For later use, we need to consider several other supergroup functors:

Definition 4.5. Let G_V be as above. We define the full subfunctors T_V , G_0 , G_0^{\pm} , G_0^- , G_0^+ , $G_{0^+}^{\pm}$ and G^{\pm} of G_V — still from $(\text{salg})_{\mathbb{k}}$ to (groups) — as given on objects, for all $A \in (\text{salg})_{\mathbb{k}}$, by

$$T_V(A) := \left\langle h_H(A) \mid H \in \mathfrak{h}_{\mathbb{Z}} \right\rangle = \left\langle h_i(A) \mid i = 1, \dots, r \right\rangle$$

$$G_0(A) := \left\langle h_i(A), x_{\tilde{\alpha}}(A) \right\rangle_{\substack{\tilde{\alpha} \in \tilde{\Delta}_0 \\ i=1, \dots, r;}}, \quad G_0^{\pm}(A) := \left\langle h_i(A), x_{\tilde{\alpha}}(A) \right\rangle_{\substack{\tilde{\alpha} \in \tilde{\Delta}_0^{\pm} \\ i=1, \dots, r;}}$$

$$\begin{aligned}
G_{\bar{0}}(A) &:= \left\langle h_i(A), x_{\tilde{\alpha}}(A) \right\rangle_{\substack{\tilde{\alpha} \in \tilde{\Delta}_0 \\ i=1, \dots, r;}} \quad , \quad G_0^{\pm}(A) := \left\langle h_i(A), x_{\tilde{\alpha}}(A) \right\rangle_{\substack{\tilde{\alpha} \in \tilde{\Delta}_0^{\pm} \\ i=1, \dots, r;}} \\
G_{\bar{0}\uparrow}(A) &:= \left\langle x_{\tilde{\alpha}}(A) \mid \tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow} \right\rangle \quad , \quad G_{\bar{0}\uparrow}^{\pm}(A) := \left\langle x_{\tilde{\alpha}}(A) \mid \tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}^{\pm} \right\rangle \\
G^{\pm}(A) &:= \left\langle x_{\tilde{\alpha}}(A) \mid \tilde{\alpha} \in \tilde{\Delta}^{\pm} \right\rangle = \langle G_0^{\pm}(A), G_1^{\pm}(A) \rangle
\end{aligned}$$

By definition $T_V, G_0, G_0^{\pm}, G_{\bar{0}}, G_{\bar{0}}^{\pm}, G_{\bar{0}\uparrow}, G_{\bar{0}\uparrow}^{\pm}, G^{\pm}, G_V$ are subgroup functors of the functor $\mathrm{GL}(V_{\mathbb{k}})$, with obvious mutual inclusions. As $\mathrm{GL}(V_{\mathbb{k}})$ is a *sheaf* (in the sense of category theory, cf. [9], Appendix), these subfunctors are *presheaves*. This implies that we can take their sheafification — with respect to the Zariski topology in $(\mathrm{salg})_{\mathbb{k}}$ — so next definition makes sense:

Definition 4.6. Consider on $(\mathrm{salg})_{\mathbb{k}}$ the Zariski topology, with respect to which $(\mathrm{salg})_{\mathbb{k}}$ itself is a site. We call *Cartan (type) supergroup*, associated with \mathfrak{g} and V , the sheafification \mathbf{G}_V of G_V (with respect to the Zariski topology). In particular, $\mathbf{G}_V: (\mathrm{salg})_{\mathbb{k}} \rightarrow (\text{groups})$ is a sheaf functor such that $\mathbf{G}_V(A) = G_V(A)$ when $A \in (\mathrm{salg})_{\mathbb{k}}$ is *local* — i.e., it has a unique maximal homogeneous ideal.

Similarly, by $\mathbf{T}_V, \mathbf{G}_0, \mathbf{G}_0^{\pm}, \mathbf{G}_{\bar{0}\uparrow}, \mathbf{G}_{\bar{0}\uparrow}^{\pm}, \mathbf{G}_{\bar{0}}, \mathbf{G}_{\bar{0}}^{\pm}$ and \mathbf{G}^{\pm} we shall denote the sheafification respectively of $T_V, G_0, G_0^{\pm}, G_{\bar{0}\uparrow}, G_{\bar{0}\uparrow}^{\pm}, G_{\bar{0}}, G_{\bar{0}}^{\pm}$ and G^{\pm} .

Remarks 4.7.

(a) The functors \mathbf{G}_V will eventually prove to be the “affine algebraic supergroups of Cartan type” which are our main object of interest. Later on, we shall prove that they are indeed representable, so they are *affine supergroups*, and algebraic, with \mathfrak{g}_V as tangent Lie superalgebra.

(b) By definition, the functors $T_V, G_0, G_0^{\pm}, G_{\bar{0}}, G_{\bar{0}}^{\pm}, G_{\bar{0}\uparrow}, G_{\bar{0}\uparrow}^{\pm}$ and their sheafifications are all supergroup functors which factor through $(\mathrm{alg}) = (\mathrm{alg})_{\mathbb{k}}$, the category of commutative, unital \mathbb{k} -algebras: thus they pertain to the domain of “classical” (i.e. “non super”) algebraic geometry.

(c) We shall see later (cf. Remark 4.25) that the functor \mathbf{G}_V can also be defined by saying that $\mathbf{G}_V(A)$, for $A \in (\mathrm{salg})_{\mathbb{k}}$, is the subgroup of $\mathrm{GL}(V_{\mathbb{k}})(A)$ generated by $\mathbf{G}_0(A)$ and the one-parameter subgroups $x_{\tilde{\alpha}}(A)$ with $\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}_0$. A similar remark holds true for some of the subgroups of \mathbf{G}_V .

(d) By definition G_V and \mathbf{G}_V — and all their supersubgroups considered above — are supersubgroups of $\mathrm{GL}(V)$. As the latter identifies with $\mathrm{GL}(V_{\bullet})$ — cf. Examples 2.6(b) — we can also think of G_V and \mathbf{G}_V (and their supersubgroups) as supersubgroups of $\mathrm{GL}(V_{\bullet})$.

In all our analysis hereafter, the key tool will be given by the commutation relations among the generators of our supergroups: these are detailed in the next lemma. As a matter of notation, when Γ is any group and $g, h \in \Gamma$ we denote by $(g, h) := g h g^{-1} h^{-1}$ their commutator in Γ .

Lemma 4.8. Let $A \in (\mathrm{salg})_{\mathbb{k}}$ be fixed.

(a) Let $\alpha, \beta \in \Delta$ with $\alpha + \beta \neq 0$; set $\Delta_{\alpha, \beta} := \Delta \cap (\mathbb{N}_+ \alpha + \mathbb{N}_+ \beta)$. Then, for all $0 \leq i \leq \mu(\alpha)$, $0 \leq j \leq \mu(\beta)$, and $\gamma \in \Delta_{\alpha, \beta}$, $0 \leq t \leq \mu(\gamma)$, there exist $c_{\gamma; t}^{k, h} \in \mathbb{Z}$ such that

$$(x_{\alpha, k}(\mathbf{p}), x_{\beta, h}(\mathbf{q})) = \prod_{\substack{\gamma = r\alpha + s\beta \in \Delta_{\alpha, \beta} \\ 0 \leq t \leq \mu(\gamma)}} x_{\gamma, t}(c_{\gamma; t}^{k, h} \mathbf{p}^r \mathbf{q}^s) \quad (4.2)$$

for any $\mathbf{p}, \mathbf{q} \in A_{\bar{0}} \cup A_{\bar{1}}$ (notation as in Definition 4.2), where the factors in right-hand side commute with one another. In particular (notation of Definition 3.1) we have the following special cases:

(a.1) assume $\alpha + \beta \notin (\Delta \cup \{0\})$, and $\mathbf{p}, \mathbf{q} \in A_{\bar{0}} \cup A_{\bar{1}}$ (with suitable parity): then

$$(x_{\alpha, k}(\mathbf{p}), x_{\beta, h}(\mathbf{q})) = 1$$

(a.2) assume $\alpha + \beta \in \Delta$, $\alpha \notin \Delta_0$, $\beta \notin \Delta_0$, and $\mathbf{p}, \mathbf{q} \in A_{\bar{0}} \cup A_{\bar{1}}$ (with suitable parity): then

$$(x_{\alpha, k}(\mathbf{p}), x_{\beta, h}(\mathbf{q})) = \prod_{t=1}^{\mu(\alpha+\beta)} x_{\alpha+\beta, t}((-1)^{p(\mathbf{p})p(\mathbf{q})} c_{\alpha, k}^{\beta, h}(t) \mathbf{p} \mathbf{q})$$

where all factors in the right-hand product do commute with each other;

(a.3) assume $\alpha + \beta \in \Delta$, $\alpha \in \Delta_0$, $\beta \notin \Delta_0$, and $\mathbf{p}, \mathbf{q} \in A_{\bar{0}} \cup A_{\bar{1}}$ (with suitable parity): then

$$(x_{\alpha,k}(\mathbf{p}), x_{\beta,h}(\mathbf{q})) = x_{2\alpha+\beta,t'}(\pm \mathbf{p}^2 \mathbf{q}) \cdot \prod_{t=1}^{\mu(\alpha+\beta)} x_{\alpha+\beta,t}(\pm c_{\alpha,k}^{\beta,h}(t) \mathbf{p} \mathbf{q})$$

(for an index t' given as in part (g) of Definition 3.1 if $2\alpha + \beta \in \Delta$, and $x_{2\alpha+\beta,t'}(\pm \mathbf{p}^2 \mathbf{q}) := 1$ if $2\alpha + \beta \notin \Delta$) where all factors on the right-hand side commute with each other.

(a.4) assume $\alpha, \beta \in \Delta_0$, and $p, q \in A_{\bar{0}}$: then

$$(x_{\alpha,1}(p), x_{\beta,1}(q)) = \prod_{\substack{\gamma=r\alpha+s\beta \in \Delta_{\alpha,\beta} \\ 0 \leq t \leq \mu(\gamma)}} x_{\alpha+\beta,1}(c_{\alpha,1}^{\beta,1}(1) p q) \in G_0(A)$$

(b) Let $\gamma \in \Delta_{-1}$, let $0 \leq j \leq \mu(-\gamma) = n-1$, and let $\vartheta, \eta \in A_{\bar{1}}$. Then

$$(x_{\gamma,1}(\vartheta), x_{-\gamma,j}(\eta)) = (1 \mp \vartheta \eta H_{\sigma_{\gamma}(j)}) = h_{H_{\sigma_{\gamma}(j)}}(1 \mp \vartheta \eta) \in G_0(A)$$

(c) Let $\alpha \in \Delta$, $0 \leq i \leq \mu(\alpha)$, $H \in \mathfrak{h}_{\mathbb{Z}}$, $u \in U(A_{\bar{0}})$, $\mathbf{u} \in A_{\bar{0}} \cup A_{\bar{1}}$ (with suitable parity). Then

$$h_H(u) x_{\alpha,i}(\mathbf{u}) h_H(u)^{-1} = x_{\alpha,i}(u^{\alpha(H)} \mathbf{u}) \in G_{p(\alpha)}(A)$$

where $p(\alpha) := \bar{s} \in \mathbb{Z}_2$, by definition, if and only if $\alpha \in \Delta_{\bar{s}}$.

Proof. (a) The proof follows from a direct analysis, through formal computations, just like in the classical case of Chevalley groups, which is treated in [17], §3, Lemma 15. We shall carry it on by looking at the general case, and later specializing to the special ones.

First of all, fix notation $X := \mathbf{p} X_{\alpha,k}$, $Y := \mathbf{q} X_{\beta,h}$. Recall that any (additive) one-parameter supersubgroup can be expressed by a formal exponential: so $x_{\alpha,k}(\mathbf{p}) := \sum_{m=0}^{+\infty} (\mathbf{p} X_{\alpha,k})^m / m! := \sum_{r=0}^{+\infty} X^r / r! = \exp(X)$, and $x_{\beta,h}(\mathbf{q}) := \sum_{s=0}^{+\infty} Y^s / s! = \exp(Y)$. Now, formal calculation gives

$$\begin{aligned} (x_{\alpha,k}(\mathbf{p}), x_{\beta,h}(\mathbf{q})) &= \text{Ad}(x_{\alpha,k}(\mathbf{p})) \left(\sum_{s=0}^{+\infty} Y^s / s! \right) \cdot x_{\beta,h}(\mathbf{q})^{-1} = \sum_{s=0}^{+\infty} \left(\text{Ad}(x_{\alpha,k}(\mathbf{p}))(Y) \right)^s / s! \cdot x_{\beta,h}(\mathbf{q})^{-1} = \\ &= \sum_{s=0}^{+\infty} \left(\text{Ad}(\exp(X))(Y) \right)^s / s! \cdot x_{\beta,h}(\mathbf{q})^{-1} = \sum_{s=0}^{+\infty} \left(\exp(\text{ad}(X))(Y) \right)^s / s! \cdot x_{\beta,h}(\mathbf{q})^{-1} \end{aligned}$$

where in the last step we used the (formal) identity $\text{Ad} \circ \exp = \exp \circ \text{ad}$. Now, moving on we get

$$\exp(\text{ad}(X))(Y) = \sum_{r=0}^{+\infty} \text{ad}(X)^r(Y) / r! = Y + [X, Y] + [X, [X, Y]] / 2 \quad (4.3)$$

because $\text{ad}(X)^r = 0$ for all $r > 2$ by Proposition 2.21(b).

As a consequence, if $\alpha + \beta \notin (\Delta \cup \{0\})$ we have $[X, Y] \in \mathfrak{g}_{\alpha+\beta}(A) = \{0\}$, hence (4.3) reads $\exp(\text{ad}(X))(Y) = Y$. Then the above analysis proves (a.1), since it yields

$$(x_{\alpha,k}(\mathbf{p}), x_{\beta,h}(\mathbf{q})) = \sum_{s=0}^{+\infty} Y^s / s! \cdot x_{\beta,h}(\mathbf{q})^{-1} = x_{\beta,h}(\mathbf{q}) \cdot x_{\beta,h}(\mathbf{q})^{-1} = 1$$

Now assume $\alpha + \beta \in \Delta$ but $\alpha, \beta \notin \Delta_0$. Then $[X, [X, Y]] = 0$ by Proposition 2.21(b) if $\alpha \in \Delta_{\bar{0}}$, and by $\mathbf{p}^2 = 0$ if $\alpha \in \Delta_{\bar{1}}$; thus (4.3) reads $\exp(\text{ad}(X))(Y) = Y + [X, Y]$. Similarly $[Y, [X, Y]] = 0$, so the summands Y and $[X, Y]$ commute with each other: thus our analysis gives

$$\begin{aligned} (x_{\alpha,k}(\mathbf{p}), x_{\beta,h}(\mathbf{q})) &= \sum_{s=0}^{+\infty} (Y + [X, Y])^s / s! \cdot x_{\beta,h}(\mathbf{q})^{-1} = \exp(Y + [X, Y]) \cdot x_{\beta,h}(\mathbf{q})^{-1} = \\ &= \exp(Y) \cdot \exp([X, Y]) \cdot x_{\beta,h}(\mathbf{q})^{-1} = \exp([X, Y]) \cdot \exp(Y) \cdot x_{\beta,h}(\mathbf{q})^{-1} = \exp([X, Y]) \end{aligned}$$

since $x_{\beta,h}(\mathbf{q}) = \exp(Y)$. Now $[X, Y] = [\mathbf{p} X_{\alpha,k}, \mathbf{q} X_{\beta,h}] = (-1)^{p(\mathbf{p})p(\mathbf{q})} \sum_{t=1}^{\mu(\alpha+\beta)} \mathbf{p} \mathbf{q} c_{\alpha,k}^{\beta,h}(t) X_{\alpha+\beta,t}$ by Definition 3.1(f), and the summands in the last term all commute with each other. Indeed, in all cases except for $\mathfrak{g} = \tilde{S}(n)$, $\alpha + \beta = -\varepsilon_i$, or $\mathfrak{g} = H(2r+1)$, $\alpha + \beta \in (2\mathbb{N}+1)\delta$, this holds because Proposition 2.21(a) give $[X_{\alpha+\beta,t'}, X_{\alpha+\beta,t''}] \in [\mathfrak{g}_{\alpha+\beta}, \mathfrak{g}_{\alpha+\beta}] = \{0\}$. In the remaining cases instead, the root $\alpha + \beta$ is odd, hence either α or β is odd as well, thus $\mathbf{p} \in A_{\bar{1}}$ or $\mathbf{q} \in A_{\bar{1}}$: therefore $[\mathbf{p} \mathbf{q} X_{\alpha+\beta,t'}, \mathbf{p} \mathbf{q} X_{\alpha+\beta,t''}] = 0$ just because $(\mathbf{p} \mathbf{q})^2 = \pm \mathbf{p}^2 \mathbf{q}^2 = 0$. The outcome is

$$\begin{aligned}
(x_{\alpha,k}(\mathbf{p}), x_{\beta,h}(\mathbf{q})) &= \exp([X, Y]) = \exp\left((-1)^{p(\mathbf{p})p(\mathbf{q})} \mathbf{p} \mathbf{q} \cdot \sum_{t=1}^{\mu(\alpha+\beta)} c_{\alpha,k}^{\beta,h}(t) X_{\alpha+\beta,t}\right) = \\
&= \prod_{t=1}^{\mu(\alpha+\beta)} \exp\left((-1)^{p(\mathbf{p})p(\mathbf{q})} \mathbf{p} \mathbf{q} \cdot c_{\alpha,k}^{\beta,h}(t) X_{\alpha+\beta,t}\right) = \prod_{t=1}^{\mu(\alpha+\beta)} x_{\alpha+\beta,t}\left((-1)^{p(\mathbf{p})p(\mathbf{q})} \mathbf{p} \mathbf{q} \cdot c_{\alpha,k}^{\beta,h}(t)\right)
\end{aligned}$$

with all factors in the last product which commute among themselves. This proves (a.2).

Finally, assume that $\alpha + \beta \in \Delta$ and $\alpha \in \Delta_0$, $\beta \notin \Delta_0$ (N.B.: the case $\alpha \notin \Delta_0$, $\beta \in \Delta_0$ is symmetric, hence we drop it). Just like before, we find $[Y, [Y, X]] = 0$; therefore $[Y, [X, Y]] = \pm[Y, [Y, X]] = 0$, then also $[Y, [X, [X, Y]]] = \pm[[Y, X], [Y, X]] \pm [X, [Y, [Y, X]]] = 0$ — by the super-Leibnitz' rule, and taking into account the identity $[[Y, X], [Y, X]] = 0$ inside $\mathfrak{g}(A) := \mathfrak{g}_0 \otimes_{\mathbb{K}} A_0 \oplus \mathfrak{g}_1 \otimes_{\mathbb{K}} A_1$, which is a Lie algebra — and finally (again by super-Leibnitz' rule) $[[X, Y], [X, [X, Y]]] = \pm[[X, [X, [X, Y]]], Y] \pm [X, [Y, [X, [X, Y]]]] = \pm[0, Y] \pm [X, 0] = 0$ by Proposition 2.21(a), for the first summand, and by $[Y, [X, [X, Y]]] = 0$ just proved, for the second. This means that the three summands in right-hand side of (4.3) do commute with each other; thus

$$\begin{aligned}
(x_{\alpha,k}(\mathbf{p}), x_{\beta,h}(\mathbf{q})) &= \exp\left(Y + [X, Y] + [X, [X, Y]]/2\right) \cdot x_{\beta,h}(\mathbf{q})^{-1} = \\
&= \exp\left([X, [X, Y]]/2\right) \cdot \exp([X, Y]) \cdot \exp(Y) \cdot x_{\beta,h}(\mathbf{q})^{-1} = \exp\left([X, [X, Y]]/2\right) \cdot \exp([X, Y])
\end{aligned}$$

because $x_{\beta,h}(\mathbf{q}) = \exp(Y)$. As before, $[X, Y] = (-1)^{p(\mathbf{p})p(\mathbf{q})} \mathbf{p} \mathbf{q} \cdot \sum_{t=1}^{\mu(\alpha+\beta)} c_{\alpha,k}^{\beta,h}(t) X_{\alpha+\beta,t}$ with the summands in the last sum which commute with each other; similarly, we expand $[X, [X, Y]]$ as $[X, [X, Y]] = [\mathbf{p} X_{\alpha,1}, [\mathbf{p} X_{\alpha,1}, \mathbf{q} Y_{\beta,h}]] = (-1)^{p(\mathbf{p})(p(\mathbf{p})+p(\mathbf{q}))} \mathbf{p}^2 \mathbf{q} [X_{\alpha,k}, [X_{\alpha,k}, Y_{\beta,h}]]$. Now, if $\alpha \notin \Delta_0$ Proposition 2.21(b) gives $[X_{\alpha,k}, [X_{\alpha,k}, Y_{\beta,h}]] = 0$, hence $[X, [X, Y]] = 0$. If instead $\alpha \in \Delta_0$, then ($k=1$ and) either $[X, [X, Y]] = \mathbf{p}^2 \mathbf{q} [X_{\alpha,1}, [X_{\alpha,1}, Y_{\beta,1}]] = \pm \mathbf{p}^2 \mathbf{q} 2 X_{2\alpha+\beta,t'}$ or $[X, [X, Y]] = 0$, by Definition 3.1(g) — for some t' as therein. Note also that $[X, [X, Y]]$ commutes with each summand in the expansion $[X, Y] = (-1)^{p(\mathbf{p})p(\mathbf{q})} \sum_{t=1}^{\mu(\alpha+\beta)} \mathbf{p} \mathbf{q} \cdot c_{\alpha,k}^{\beta,h}(t) X_{\alpha+\beta,t}$; indeed, this occurs because $[[X, [X, Y]], X_{\alpha+\beta,t}] \in \mathfrak{g}_{3\alpha+2\beta}(A)$, and direct (straightforward) inspection shows that $3\alpha+2\beta \notin \Delta$ (having $\alpha+\beta \in \Delta$ and $\alpha \in \Delta_0$, by assumption). So we find

$$\begin{aligned}
(x_{\alpha,1}(\mathbf{p}), x_{\beta,h}(\mathbf{q})) &= \exp\left([X, [X, Y]]/2\right) \cdot \exp([X, Y]) = \\
&= \exp(\pm \mathbf{p}^2 \mathbf{q} X_{2\alpha+\beta,t'}) \prod_{t=1}^{\mu(\alpha+\beta)} \exp(\mathbf{p} \mathbf{q} c_{\alpha,1}^{\beta,h}(t) X_{\alpha+\beta,t}) = x_{2\alpha+\beta,t'}(\pm \mathbf{p}^2 \mathbf{q}) \prod_{t=1}^{\mu(\alpha+\beta)} x_{\alpha+\beta,t}(\pm c_{\alpha,1}^{\beta,h}(t) \mathbf{p} \mathbf{q})
\end{aligned}$$

(with $x_{2\alpha+\beta,t'}(\pm \mathbf{p}^2 \mathbf{q}) := 1$ if $2\alpha+\beta \notin \Delta$) with all factors pairwise commuting, so (a.3) is proved.

The very last case to consider is (a.4), which is a classical result: see [17], §3, Lemma 15.

(b) The same arguments used for (a) give also $(x_{\gamma,1}(\vartheta), x_{-\gamma,j}(\eta)) = 1 - \vartheta \eta [X_{\gamma,1}, X_{-\gamma,j}]$. Then Definition 3.1(d) gives $[X_{\gamma,1}, X_{-\gamma,j}] = \pm H_{\sigma_{\gamma}(j)}$. Plugging this into the previous formula, and noting that $(\vartheta \eta)^n = 0$ for all $n > 1$, we get exactly (b).

(c) Let $v_\mu \in M_\mu := (M \cap V_\mu)$ be any weight vector in the admissible lattice M of V used to define G_V . We show now that $h_H(u) x_{\alpha,i}(\mathbf{u}) h_H(u)^{-1}$ and $x_{\alpha,i}(u^{\alpha(H)} \mathbf{u})$ acts on the same way on v_μ : taking μ and v_μ arbitrarily, this is enough to prove claim (c). Direct computation gives

$$\begin{aligned}
(h_H(u) x_{\alpha,i}(\mathbf{u}) h_H(u)^{-1})(v_\mu) &= u^{\mu(H)} \cdot h_H(u) \left(\sum_{n=0}^{+\infty} \text{ad}(\mathbf{u} X_{\alpha,i})^n(v_\mu)/n! \right) = \\
&= u^{-\mu(H)} \cdot \sum_{n=0}^{+\infty} \frac{1}{n!} h_H(u) (\text{ad}(\mathbf{u} X_{\alpha,i})^n(v_\mu)) = u^{-\mu(H)} \sum_{n=0}^{+\infty} \frac{1}{n!} u^{(\mu+n\alpha)(H)} \text{ad}(\mathbf{u} X_{\alpha,i})^n(v_\mu) = \\
&= u^{-\mu(H)} u^{\mu(H)} \sum_{n=0}^{+\infty} \frac{1}{n!} \text{ad}(u^{\alpha(H)} \mathbf{u} X_{\alpha,i})^n(v_\mu) = \exp(\text{ad}(u^{\alpha(H)} \mathbf{u} X_{\alpha,i}))(v_\mu) = x_{\alpha,i}(u^{\alpha(H)} \mathbf{u})(v_\mu)
\end{aligned}$$

which is exactly what we needed. \square

4.3 The even part \mathbf{G}_0 of \mathbf{G}_V

Our definition of the supergroup \mathbf{G}_V does not imply (at first sight) that \mathbf{G}_V be *representable*. In order to prove that, we need to know how the “even part” \mathbf{G}_0 of \mathbf{G}_V looks like.

Proposition 4.9. *The functor \mathbf{G}_0 is representable, hence — as it factors through $(\text{alg})_{\mathbb{k}}$ — it is an affine group-scheme; moreover, it is also algebraic. More precisely, we have natural isomorphisms $\mathbf{G}_0 \cong \mathbf{Ch}_V$, where $\mathbf{Ch}_V : (\text{alg})_{\mathbb{k}} \rightarrow (\text{groups})$ is the standard (affine, algebraic) group functor associated with \mathfrak{g}_0 and with the \mathfrak{g}_0 -module V by the classical Chevalley-Demazure construction.*

Proof. This is just a consequence of the very definitions. Indeed, in terms of category theory (cf. for instance [19]), the category $(\text{alg}) = (\text{alg})_{\mathbb{k}}$ is a site and both the functors \mathbf{G}_0 and \mathbf{Ch}_V are sheaves.

Moreover, by construction there exist natural transformations $G_0 \xrightarrow{\alpha} \mathbf{G}_0$, $G_0 \xrightarrow{\beta} \mathbf{Ch}_V$, and G_0 coincides with \mathbf{Ch}_V via β on local algebras, that is $\beta(G_0(R)) = \mathbf{Ch}_V(R)$ for any $R \in \text{Ob}(\text{alg})$ which is local (this follows from §3.5.3 in [6] and Corollaire 5.7.6 in [5]). As \mathbf{G}_0 by definition is the sheafification of G_0 , the universal property characterizing the sheafification yields $\mathbf{G}_0 \cong \mathbf{Ch}_V$. \square

As a second step, we have the following result for some (classical) subgroup functors of $\mathbf{G}_{\bar{0}}$:

Proposition 4.10. *Fix any total order \preceq in $\tilde{\Delta}_{\bar{0}\uparrow}$. Then we have:*

- (a) $G_{\bar{0}\uparrow}(A) = \prod_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}} x_{\tilde{\alpha}}(A)$ for all $A \in (\text{salg})_{\mathbb{k}}$, the product being ordered according to \preceq ;
- (b) $G_{\bar{0}\uparrow} \trianglelefteq G_{\bar{0}}$ and $\mathbf{G}_{\bar{0}\uparrow} \trianglelefteq \mathbf{G}_{\bar{0}}$, where \trianglelefteq stands for “normal subgroup functor”;
- (c) $G_{\bar{0}} = G_0 \cdot G_{\bar{0}\uparrow} = G_{\bar{0}\uparrow} \cdot G_0$ and $\mathbf{G}_{\bar{0}} = \mathbf{G}_0 \cdot \mathbf{G}_{\bar{0}\uparrow} = \mathbf{G}_{\bar{0}\uparrow} \cdot \mathbf{G}_0$. In particular, $\mathbf{G}_{\bar{0}}$ is a closed subgroup of $\text{GL}(V)$, hence it is in turn (on its own) an affine algebraic group.
- (d) the group functors $G_{\bar{0}\uparrow}$ and $\mathbf{G}_{\bar{0}\uparrow}$ are both unipotent.

Proof. (a) The formulas for commutators in Lemma 4.8 imply that any (unordered) product of several factors $x_{\tilde{\alpha}}(t_{\tilde{\alpha}})$ with $\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}$ can be reordered. In fact, whenever we have a couple of consecutive unordered factors, say $x_{\tilde{\alpha}_1}(t_{\tilde{\alpha}_1}) x_{\tilde{\alpha}_2}(t_{\tilde{\alpha}_2})$, we can re-write their product as

$$x_{\tilde{\alpha}_1}(t_{\tilde{\alpha}_1}) x_{\tilde{\alpha}_2}(t_{\tilde{\alpha}_2}) = (x_{\tilde{\alpha}_1}(t_{\tilde{\alpha}_1}), x_{\tilde{\alpha}_2}(t_{\tilde{\alpha}_2})) \cdot x_{\tilde{\alpha}_2}(t_{\tilde{\alpha}_2}) x_{\tilde{\alpha}_1}(t_{\tilde{\alpha}_1})$$

Then formula (4.2) for $(x_{\tilde{\alpha}_1}(t_{\tilde{\alpha}_1}), x_{\tilde{\alpha}_2}(t_{\tilde{\alpha}_2}))$ tells us that the commutator is either 1, or a product of several $x_{\tilde{\alpha}}(t_{\tilde{\alpha}})$ such that $ht(\pi(\tilde{\alpha})) \geq ht(\pi(\tilde{\alpha}_1))$, $ht(\pi(\tilde{\alpha})) \geq ht(\pi(\tilde{\alpha}_2))$ (cf. §2.18). Therefore, we can iterate this process in order to commute all unordered pairs of factors, up to (possibly) introducing new factors. However, the above shows that these new factors, if any, will be attached to roots with greater height: as the height is bounded from above, we shall end up with trivial new factors, i.e. after finitely many steps all pairs can be reordered without introducing new factors.

As a consequence, the multiplication map $\times_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}} x_{\tilde{\alpha}}(A) \rightarrow G_V(A)$ — the product on left-hand side being ordered — yields a surjection onto $G_{\bar{0}\uparrow}(A)$, realized as $G_{\bar{0}\uparrow}(A) = \prod_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}} x_{\tilde{\alpha}}(A)$.

(b) Again Lemma 4.8 gives that $\prod_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}} x_{\tilde{\alpha}}(A)$ is normalized by G_0 ; then by (a), we deduce that $G_{\bar{0}\uparrow} \trianglelefteq G_{\bar{0}}$, whence $\mathbf{G}_{\bar{0}\uparrow} \trianglelefteq \mathbf{G}_{\bar{0}}$ follows too.

(c) This follows easily by construction, namely from $G_0 \leq G_{\bar{0}}$, $\mathbf{G}_0 \leq \mathbf{G}_{\bar{0}}$ and $G_{\bar{0}} = \langle G_0, G_{\bar{0}\uparrow} \rangle$, along with (b). By classical theory of algebraic groups, as \mathbf{G}_0 and $\mathbf{G}_{\bar{0}\uparrow}$ are closed subgroups of $\text{GL}(V)$ and \mathbf{G}_0 normalizes $\mathbf{G}_{\bar{0}\uparrow}$ one argues that the last part of claim (c) holds too.

(d) This follows from the classical theory of affine group-schemes, because we have embeddings $G_{\bar{0}\uparrow} \leq \mathbf{G}_{\bar{0}\uparrow} \leq \text{GL}(V)$, and the tangent Lie algebra of $\mathbf{G}_{\bar{0}\uparrow}$, i.e. $\text{Lie}(\mathbf{G}_{\bar{0}\uparrow}) = \mathfrak{g}_{\bar{0}\uparrow}$, is nilpotent. \square

The same (type of) arguments proves also the following:

Proposition 4.11. *Fix any total order \preceq in $\tilde{\Delta}_{\bar{0}\uparrow}^{\pm}$. Then we have:*

- (a) $G_{\bar{0}\uparrow}^{\pm}(A) = \prod_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}^{\pm}} x_{\tilde{\alpha}}(A)$ for all $A \in (\text{salg})_{\mathbb{k}}$, the product being ordered according to \preceq ;
- (b) $G_{\bar{0}\uparrow}^{\pm} \trianglelefteq G_{\bar{0}}^{\pm}$ and $\mathbf{G}_{\bar{0}\uparrow}^{\pm} \trianglelefteq \mathbf{G}_{\bar{0}}^{\pm}$.
- (c) $G_{\bar{0}}^{\pm} = G_0^{\pm} \cdot G_{\bar{0}\uparrow}^{\pm}$ and $\mathbf{G}_{\bar{0}}^{\pm} = \mathbf{G}_0^{\pm} \cdot \mathbf{G}_{\bar{0}\uparrow}^{\pm}$.
- (d) the group functors $G_{\bar{0}\uparrow}^{\pm}$ and $\mathbf{G}_{\bar{0}\uparrow}^{\pm}$ are both unipotent.

Proposition 4.10 and Proposition 4.11 can also be improved as follows:

Proposition 4.12. *The group product yields group-functor isomorphisms*

$$G_{\bar{0}} \cong G_0 \ltimes G_{\bar{0}\uparrow}, \quad G_{\bar{0}}^{\pm} \cong G_0^{\pm} \ltimes G_{\bar{0}\uparrow}^{\pm} \quad \text{and} \quad \mathbf{G}_{\bar{0}} \cong \mathbf{G}_0 \ltimes \mathbf{G}_{\bar{0}\uparrow}, \quad \mathbf{G}_{\bar{0}}^{\pm} \cong \mathbf{G}_0^{\pm} \ltimes \mathbf{G}_{\bar{0}\uparrow}^{\pm}.$$

Proof. The right-hand side pair of isomorphisms clearly follows from the left-hand side one. As for the latter, we have to prove that $G_{\bar{0}}(A) \cong G_0(A) \ltimes G_{\bar{0}\uparrow}(A)$ and $G_{\bar{0}}^{\pm}(A) \cong G_0^{\pm}(A) \ltimes G_{\bar{0}\uparrow}^{\pm}(A)$ for every $A \in (\text{salg})_{\mathbb{k}}$, and also to show that these isomorphisms are functorial in A : this second part will be trivial, so we cope just with the first one. Actually, we prove $G_{\bar{0}}(A) \cong G_0(A) \ltimes G_{\bar{0}\uparrow}(A)$ only, for the proof of $G_{\bar{0}}^{\pm}(A) \cong G_0^{\pm}(A) \ltimes G_{\bar{0}\uparrow}^{\pm}(A)$ is quite the same.

For any $A \in (\text{salg})_{\mathbb{k}}$, we know by Proposition 4.10 that $G_0(A) \leq G_{\bar{0}}(A)$, $G_{\bar{0}\uparrow}(A) \trianglelefteq G_{\bar{0}}(A)$ and $G_{\bar{0}}(A) = G_0(A) \cdot G_{\bar{0}\uparrow}(A)$. Thus we are only left to prove $G_0(A) \cap G_{\bar{0}\uparrow}(A) = \{e_{G_V}\}$.

Let $A \in (\text{salg})_{\mathbb{k}}$, and let $g \in G_0(A) \cap G_{\bar{0}\uparrow}(A)$: then $g = g_0 \in G_0(A)$ and $g = g_{\uparrow} \in G_{\bar{0}\uparrow}(A)$, in particular $g_0 = g_{\uparrow}$. Now let V be the \mathfrak{g} -module we use to define G_V , \mathbf{G}_V , etc., splitting as $V = \bigoplus_{\mu} V_{\mu}$ into direct sum of weight spaces. All root vectors of \mathfrak{g} map weight spaces into weight spaces, namely $X_{\tilde{\eta}} \cdot V_{\mu} \subseteq V_{\mu+\eta}$ if $X_{\tilde{\eta}} \in \mathfrak{g}_{\eta}$ (for each root η and every weight μ). This implies that, for all weights μ and $v_{\mu} \in V_{\mu}(A)$, $H \in \mathfrak{h}_{\mathbb{Z}}$, $\tilde{\alpha} \in \tilde{\Delta}$ and $\alpha := \pi(\tilde{\alpha})$, one has (notation of §2.18)

$$h_H(A) \cdot v_{\mu} \in A_{\bar{0}} v_{\mu} \subseteq V_{\mu}(A), \quad x_{\tilde{\alpha}}(A) \cdot v_{\mu} \in v_{\mu} + \left(\bigoplus_{n \in \mathbb{N}_+} V_{\mu+n\alpha}(A) \right) \quad (4.4)$$

Now, by definition, $G_0(A)$ is generated by all the $h_H(A)$ and all the $x_{\tilde{\alpha}}(A)$ with $\tilde{\alpha} \in \tilde{\Delta}_0$; similarly, $G_{\bar{0}\uparrow}(A)$ is generated by all the $x_{\tilde{\alpha}}(A)$ with $\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}$. This together with (4.4) implies

$$g_0 \cdot v_{\mu} \in \bigoplus_{\beta \in \mathbb{N}\tilde{\Delta}_0} V_{\mu+\beta}(A), \quad g_{\uparrow} \cdot v_{\mu} \in v_{\mu} + \left(\bigoplus_{\gamma \in \mathbb{N}\tilde{\Delta}_{\bar{0}\uparrow} \setminus \{0\}} V_{\mu+\gamma}(A) \right) \quad (4.5)$$

for any weight μ and any $v_{\mu} \in V_{\mu}$, where $\mathbb{N}\tilde{\Delta}_0$ and $\mathbb{N}\tilde{\Delta}_{\bar{0}\uparrow}$ are the \mathbb{N} -span of $\tilde{\Delta}_0$ and of $\tilde{\Delta}_{\bar{0}\uparrow}$ respectively. Definitions give also $\mathbb{N}\tilde{\Delta}_0 \cap \mathbb{N}\tilde{\Delta}_{\bar{0}\uparrow} = \{0\}$: therefore, from (4.5) and $g_0 = g_{\uparrow}$ we infer that $g_0 \cdot v_{\mu} = v_{\mu} = g_{\uparrow} \cdot v_{\mu}$. Since μ and $v_{\mu} \in V_{\mu}(A)$ were arbitrarily chosen, and since $G_V(A)$ acts faithfully on $V(A)$, we eventually conclude that $g_0 = e_{G_V} = g_{\uparrow}$. \square

Like in the classical case of Chevalley groups, one has also the following auxiliary result:

Lemma 4.13. *Let $\tilde{\mathcal{S}} \subseteq \tilde{\Delta}$ and $\mathcal{S} := \left\{ \pi(\tilde{\alpha}) \mid \tilde{\alpha} \in \tilde{\mathcal{S}} \right\}$ (cf. §2.18). Assume that \mathcal{S} is closed, i.e. $\alpha, \beta \in \mathcal{S}$ and $\alpha + \beta \in \Delta$ imply that $\alpha + \beta \in \mathcal{S}$; assume also that $\alpha \in \mathcal{S}$ implies $-\alpha \notin \mathcal{S}$, let $G_{\tilde{\mathcal{S}}} := \left\langle x_{\tilde{\alpha}} \mid \tilde{\alpha} \in \tilde{\mathcal{S}} \right\rangle$ be the full subfunctor of G_V generated by the one-parameter subgroups indexed by the elements in $\tilde{\mathcal{S}}$, and let $\mathbf{G}_{\tilde{\mathcal{S}}}$ be the sheafification of $G_{\tilde{\mathcal{S}}}$.*

For any total order in $\tilde{\mathcal{S}}$, the group product yields scheme isomorphisms $\bigtimes_{\tilde{\alpha} \in \tilde{\mathcal{S}}} x_{\tilde{\alpha}} \cong G_{\tilde{\mathcal{S}}}$, $\bigtimes_{\tilde{\alpha} \in \tilde{\mathcal{S}}} x_{\tilde{\alpha}} \cong \mathbf{G}_{\tilde{\mathcal{S}}}$, where the direct products on the left-hand side are ordered ones.

In particular, one has $G_{\tilde{\mathcal{S}}} = \mathbf{G}_{\tilde{\mathcal{S}}} \cong \mathbb{A}^{s_{\bar{0}}|s_{\bar{1}}}$ as superschemes, where $s_{\tilde{\mathcal{S}}} := \left| \tilde{\mathcal{S}} \cap \tilde{\Delta}_{\tilde{\mathcal{S}}} \right|$.

Proof. Each one-parameter supersubgroup $x_{\tilde{\alpha}}$ is a representable supergroup, so (as a superscheme) it is a sheaf. Any direct product of sheaves is itself a sheaf, so the left-hand side isomorphisms in the claim, once proved, implies that $G_{\tilde{\mathcal{S}}}$ is already a sheaf, so it coincides with $\mathbf{G}_{\tilde{\mathcal{S}}}$. Also, the superscheme $x_{\tilde{\eta}}$ is isomorphic to $\mathbb{A}^{1|0}$ or $\mathbb{A}^{0|1}$ according to whether $\pi(\tilde{\eta})$ is even or odd (cf. Proposition 4.3(b)), so $\bigtimes_{\tilde{\alpha} \in \tilde{\mathcal{S}}} x_{\tilde{\alpha}} \cong \mathbb{A}^{s_{\bar{0}}|s_{\bar{1}}}$ is clear. So we are left to prove the claim for $G_{\tilde{\mathcal{S}}}$.

Our task is to show that, for any $A \in (\text{salg})_{\mathbb{k}}$, the product map $\bigtimes_{\tilde{\alpha} \in \tilde{\mathcal{S}}} x_{\tilde{\alpha}}(A) \rightarrow G_{\tilde{\mathcal{S}}}(A)$ in $G_{\tilde{\mathcal{S}}}$ is a bijection: i.e., every $g \in G_{\tilde{\mathcal{S}}}$ admits a unique factorization as an ordered product $g = \prod_{\tilde{\alpha} \in \tilde{\mathcal{S}}} x_{\tilde{\alpha}}(\mathbf{t}_{\tilde{\alpha}})$ for some $\mathbf{t}_{\tilde{\alpha}} \in A_{\bar{0}} \cup A_{\bar{1}}$. This result can be found via the classical argument — cf. [17], §3, pp. 24–25 — which now works again using Lemma 4.8 as the basic ingredient. \square

A direct application of the previous lemma is the following ($\tilde{\mathcal{S}} \in \left\{ \tilde{\Delta}_{\bar{0}\uparrow}, \tilde{\Delta}_{\bar{0}}^{\pm}, \tilde{\Delta}_0^{\pm}, \tilde{\Delta}_{\bar{0}\uparrow}^{\pm}, \tilde{\Delta}^{\pm} \right\}$):

Proposition 4.14. *Fix any total order in $\tilde{\Delta}_{0\uparrow}$, in $\tilde{\Delta}_0^\pm$, in $\tilde{\Delta}_0^\pm$, in $\tilde{\Delta}_{0\uparrow}^\pm$, in $\tilde{\Delta}^\pm$. Then the group product yields scheme isomorphisms*

$$\begin{aligned} \times_{\tilde{\alpha} \in \tilde{\Delta}_{0\uparrow}} x_{\tilde{\alpha}} &\cong G_{0\uparrow} \cong \mathbf{G}_{0\uparrow}, & \times_{\tilde{\alpha} \in \tilde{\Delta}_0^\pm} x_{\tilde{\alpha}} &\cong G_0^\pm \cong \mathbf{G}_0^\pm, & \times_{\tilde{\alpha} \in \tilde{\Delta}_{0\uparrow}^\pm} x_{\tilde{\alpha}} &\cong G_0^\pm \cong \mathbf{G}_0^\pm \\ \times_{\tilde{\alpha} \in \tilde{\Delta}_{0\uparrow}^\pm} x_{\tilde{\alpha}} &\cong G_{0\uparrow}^\pm \cong \mathbf{G}_{0\uparrow}^\pm, & \times_{\tilde{\alpha} \in \tilde{\Delta}^\pm} x_{\tilde{\alpha}} &\cong G^\pm \cong \mathbf{G}^\pm \end{aligned}$$

where the direct product on the left is always ordered according to the fixed total order. In particular

$$\begin{aligned} G_{0\uparrow} &\cong \mathbf{G}_{0\uparrow} \cong \mathbb{A}^{N_{0\uparrow}|0}, & G_0^\pm &\cong \mathbf{G}_0^\pm \cong \mathbb{A}^{N_0^\pm|0}, & G_0^\pm &\cong \mathbf{G}_0^\pm \cong \mathbb{A}^{N_0|0} \\ G_{0\uparrow}^\pm &\cong \mathbf{G}_{0\uparrow}^\pm \cong \mathbb{A}^{N_{0\uparrow}^\pm|0}, & G^\pm &\cong \mathbf{G}^\pm \cong \mathbb{A}^{N^\pm|0} \end{aligned}$$

as superschemes, where $N_{0\uparrow} := |\tilde{\Delta}_{0\uparrow}|$, $N_0^\pm := |\tilde{\Delta}_0^\pm|$, $N_0 := |\tilde{\Delta}_0|$, $N_{0\uparrow}^\pm := |\tilde{\Delta}_{0\uparrow}^\pm|$, $N^\pm := |\tilde{\Delta}^\pm|$.

4.4 The functors \mathbf{G}_V as affine algebraic supergroups

In this subsection we shall show that the supergroup functors \mathbf{G}_V defined in subsection 4.2 are (the functors of points of) affine supergroups, and also algebraic. We need some more definitions:

Definition 4.15. For any $A \in (\text{salg})_{\mathbb{k}}$, we define the subsets of $G(A)$

$$G_{\bar{1}}(A) := \left\{ \prod_{i=1}^m x_{\tilde{\gamma}_i}(\vartheta_i) \right\}_{m \in \mathbb{N}, \tilde{\gamma}_i \in \tilde{\Delta}_{\bar{1}}, \vartheta_i \in A_{\bar{1}}}, \quad G_{\bar{1}}^\pm(A) := \left\{ \prod_{i=1}^m x_{\tilde{\gamma}_i}(\vartheta_i) \right\}_{m \in \mathbb{N}, \tilde{\gamma}_i \in \tilde{\Delta}_{\bar{1}}^\pm, \vartheta_i \in A_{\bar{1}}}$$

Let $N_\pm := |\tilde{\Delta}_{\bar{1}}^\pm|$ and $N := |\tilde{\Delta}_{\bar{1}}| = N_+ + N_-$, and fix total orders \preceq in $\tilde{\Delta}_{\bar{1}}^\pm$ and $\tilde{\Delta}_{\bar{1}}$: we set

$$\begin{aligned} G_{\bar{1}}^{\pm, <}(A) &:= \left\{ \prod_{i=1}^{N_\pm} x_{\tilde{\gamma}_i}(\vartheta_i) \mid \tilde{\gamma}_1 \preceq \cdots \preceq \tilde{\gamma}_{N_\pm} \in \tilde{\Delta}_{\bar{1}}^\pm, \vartheta_1, \dots, \vartheta_{N_\pm} \in A_{\bar{1}} \right\} \\ G_{\bar{1}}^<(A) &:= \left\{ \prod_{i=1}^N x_{\tilde{\gamma}_i}(\vartheta_i) \mid \tilde{\gamma}_1 \preceq \cdots \preceq \tilde{\gamma}_N \in \tilde{\Delta}_{\bar{1}}, \vartheta_1, \dots, \vartheta_N \in A_{\bar{1}} \right\} \end{aligned}$$

We use also similar notations to denote the sheafifications $\mathbf{G}_{\bar{1}}$, $\mathbf{G}_{\bar{1}}^\pm$ and $\mathbf{G}_{\bar{1}}^{\pm, <}$.

Using once more Lemma 4.8, we obtain the following factorization result for the functor G_V :

Proposition 4.16. *Let $A \in (\text{salg})_{\mathbb{k}}$. There exist set-theoretic factorizations*

$$G_V(A) = G_{\bar{0}}(A) G_{\bar{1}}(A) = G_{\bar{1}}(A) G_{\bar{0}}(A), \quad G^\pm(A) = G_0^\pm(A) G_{\bar{1}}^\pm(A) = G_{\bar{1}}^\pm(A) G_0^\pm(A)$$

Proof. The proof for $G_V(A)$ works for $G^\pm(A)$ too, so we stick to the former.

It is enough to prove either one of the equalities, say the first one. Also, it is enough to show that $G_{\bar{0}}(A) G_{\bar{1}}(A)$ is closed by multiplication: thus we must show that $g_{\bar{0}} g_{\bar{1}} \cdot g'_{\bar{0}} g'_{\bar{1}} \in G_{\bar{0}}(A) G_{\bar{1}}(A)$ for all $g_{\bar{0}}, g'_{\bar{0}} \in G_{\bar{0}}(A)$ and $g_{\bar{1}}, g'_{\bar{1}} \in G_{\bar{1}}(A)$. By the very definitions, we need only to prove that

$$(1 + \vartheta_1 X_{\tilde{\beta}_1}) \cdots (1 + \vartheta_m X_{\tilde{\beta}_m}) x_{\tilde{\alpha}}(t), \quad (1 + \vartheta_1 X_{\tilde{\beta}_1}) \cdots (1 + \vartheta_m X_{\tilde{\beta}_m}) h_\eta(u) \in G_{\bar{0}}(A) G_{\bar{1}}(A)$$

for all $m \in \mathbb{N}$, $\tilde{\beta}_1, \dots, \tilde{\beta}_m \in \tilde{\Delta}_{\bar{1}}$, $\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}}$, $\eta \in \Delta$, $\vartheta_1, \dots, \vartheta_m \in A_{\bar{1}}$, $t \in A_{\bar{0}}$ and $u \in U(A_{\bar{0}})$. But this follows by an easy induction on m , via the formulas in Lemma 4.8. \square

Carrying further on our analysis, we shall improve the above result by replacing the factor $G_{\bar{1}}$ with a factor $G_{\bar{1}}^<$. As intermediate step, this requires the following technical result:

Lemma 4.17. *Let $A \in (\text{salg})_{\mathbb{k}}$. Then — with notation of subsection 2.1 — we have*

$$\begin{aligned} G_{\bar{1}}(A) &\subseteq G_{\bar{0}}(A_1^{(2)}) G_{\bar{1}}^<(A), & G_{\bar{1}}(A) &\subseteq G_{\bar{1}}^<(A) G_{\bar{0}}(A_1^{(2)}) \\ G_{\bar{1}}^\pm(A) &\subseteq G_{\bar{0}}^\pm(A_1^{(2)}) G_{\bar{1}}^{\pm, <}(A), & G_{\bar{1}}^\pm(A) &\subseteq G_{\bar{1}}^{\pm, <}(A) G_{\bar{0}}^\pm(A_1^{(2)}) \end{aligned}$$

Proof. We deal with the first identity, the other being similar. Indeed, we prove the stronger result

$$\langle G_{\bar{1}}(A), G_{\bar{0}}(A_1^{(2)}) \rangle \subseteq G_{\bar{0}}(A_1^{(2)}) G_{\bar{1}}^{\leq}(A) \quad (4.6)$$

where $\langle G_{\bar{1}}(A), G_{\bar{0}}(A_1^{(2)}) \rangle$ is the subgroup generated by $G_{\bar{1}}(A)$ and $G_{\bar{0}}(A_1^{(2)})$.

Any element of $\langle G_{\bar{1}}(A), G_{\bar{0}}(A_1^{(2)}) \rangle$ is a product $g = g_1 g_2 \cdots g_k$ in which each factor g_i is either of type $h_{\eta_i}(u_i)$, or $x_{\tilde{\alpha}_i}(t_i)$, or $x_{\tilde{\gamma}_i}(\vartheta_i)$, with $\eta_i \in \Delta$, $\tilde{\alpha}_i \in \tilde{\Delta}_{\bar{0}}$, $\tilde{\gamma}_i \in \tilde{\Delta}_{\bar{1}}$ and $u_i \in U(A_1^{(2)})$, $t_i \in A_1^2$, $\vartheta_i \in A_{\bar{1}}$. Such a product belongs to $G_{\bar{0}}(A_1^{(2)}) G_{\bar{1}}^{\leq}(A)$ if and only if all factors indexed by the $\eta_i \in \Delta$ and by the $\tilde{\alpha}_j \in \tilde{\Delta}_{\bar{0}}$ are on the left of those indexed by the $\tilde{\gamma}_\ell \in \tilde{\Delta}_{\bar{1}}$, and moreover the latter occur in the order prescribed by \preceq . In this case, we say that the factors of g are ordered. We shall now re-write g as a product of ordered factors, by repeatedly commuting the original factors, as well as new factors which come in along this process.

As we have only a finite number of odd coefficients in the expression for g , we can assume without loss of generality that $A_{\bar{1}}$ is finitely generated as an $A_{\bar{0}}$ -module. If \overline{m} is the cardinality of any (finite) set of (odd) generators of $A_{\bar{1}}$, this implies $A_{\bar{1}}^m = \{0\}$ and $A_{\bar{1}}^{(m)} = 0$ when $m > \overline{m}$.

Let us consider two consecutive factors $g_i g_{i+1}$ in g . If they are already ordered, we are done. Otherwise, there are four possibilities:

- (1) $g_i = x_{\tilde{\gamma}_i}(\vartheta_i)$, $g_{i+1} = h_{\eta_i}(u_i)$. In this case we rewrite

$$g_i g_{i+1} = x_{\tilde{\gamma}_i}(\vartheta_i) h_{\eta_i}(u_i) = h_{\eta_i}(u_i) x_{\tilde{\gamma}_i}(\vartheta'_i)$$

with $\vartheta'_i \in A_{\bar{1}}^{m_i}$ if $\vartheta_i \in A_{\bar{1}}^{m_i}$, thanks to Lemma 4.8(c). In particular we replace a pair of unordered factors with a new pair of ordered factors. Even more, this shows that any factor of type $h_{\eta_i}(u_i)$ can be flushed to the left of our product so to give a new product of the same nature, but with all factors of type $h_{\eta_i}(u_i)$ on the left-hand side.

- (2) $g_i = x_{\tilde{\gamma}_i}(\vartheta_i)$, $g_{i+1} = x_{\tilde{\alpha}_{i+1}}(t_{i+1})$. In this case we rewrite

$$g_i g_{i+1} = g_{i+1} g_i g'_i \quad \text{with} \quad g'_i := (g_i^{-1}, g_{i+1}^{-1}) = (x_{\tilde{\gamma}_i}(-\vartheta_i), x_{\tilde{\alpha}_{i+1}}(-t_{i+1}))$$

so we replace a pair of (consecutive) unordered factors with a pair of ordered factors followed by another, new factor g'_i . However, letting $m_1, m_2 \in \mathbb{N}_+$ be such that $\vartheta_i \in A_{\bar{1}}^{m'_1}$, $t_{i+1} \in A_{\bar{1}}^{m''_2}$, by Lemma 4.8 this g'_i is a product of new factors of type $x_{\tilde{\gamma}_j}(\vartheta'_j)$ with $\vartheta'_j \in A_{\bar{1}}^{m'_j}$, $m_j \geq m'_1 + m''_2$.

- (3) $g_i = x_{\tilde{\gamma}_i}(\vartheta_i)$, $g_{i+1} = x_{\tilde{\gamma}_{i+1}}(\vartheta_{i+1})$. In this case we rewrite

$$g_i g_{i+1} = g_{i+1} g_i g'_i \quad \text{with} \quad g'_i := (g_i^{-1}, g_{i+1}^{-1}) = (x_{\tilde{\gamma}_i}(-\vartheta_i), x_{\tilde{\gamma}_{i+1}}(-\vartheta_{i+1}))$$

so we replace a pair of unordered factors with a pair of ordered ones, followed by a new factor g'_i which, again by Lemma 4.8, is a product of new factors of type $x_{\tilde{\alpha}_j}(t'_j)$ or $h_{\eta_j}(u'_j)$ with $t'_j \in A_{\bar{1}}^{m'_j}$, $u'_j \in U(A_{\bar{1}}^{(m'_j)})$, where $m_j \geq m'_1 + m''_2$ for $m'_1, m''_2 \in \mathbb{N}_+$ such that $\vartheta_i \in A_{\bar{1}}^{m'_1}$, $\vartheta_{i+1} \in A_{\bar{1}}^{m''_2}$.

- (4) $g_i = x_{\tilde{\gamma}}(\vartheta_i)$, $g_{i+1} = x_{\tilde{\gamma}}(\vartheta_{i+1})$. In this case we rewrite

$$g_i g_{i+1} = x_{\tilde{\gamma}}(\vartheta_i) x_{\tilde{\gamma}}(\vartheta_{i+1}) = x_{\tilde{\gamma}}(\vartheta_i) x_{\tilde{\gamma}}(\vartheta_{i+1}) = x_{\tilde{\gamma}}(\vartheta_i + \vartheta_{i+1}) =: g'_i$$

so we replace a pair of unordered factors with a single factor. In addition, each pair $g_{i-1} g'_i$ and $g'_i g_{i+2}$ respects or violates the ordering according to what the old pair $g_{i-1} g_i$ and $g_{i+1} g_{i+2}$ did.

Now we iterate this process: whenever we have any unordered pair of consecutive factors in the product we are working with, we perform any one of steps (1) through (4) explained above. At each step, we substitute an unordered pair with a single factor (step (4)), which does not form any more unordered pairs than the ones we had before, or with an ordered pair (steps (1)–(3)), possibly introducing new additional factors. However, any new factor is either of type $x_{\tilde{\alpha}}(t)$, with $t \in A_{\bar{1}}^m$, or of type $x_{\tilde{\gamma}}(\vartheta)$, with $\vartheta \in A_{\bar{1}}^m$, or of type $h_{\eta}(u)$, with $u \in U(A_{\bar{1}}^{(m)})$: in all cases, the values of m are (overall) strictly increasing after each iteration of this procedure. As $A_{\bar{1}}^m = \{0\}$ for $m \gg 0$, after finitely many steps such new factors are trivial, i.e. eventually all unordered (consecutive) factors will commute with each other and will be re-ordered without introducing any new factors. Thus the process stops after finitely many steps, proving (4.6). \square

A direct consequence of Proposition 4.16 and Lemma 4.17 is the following “factorization result”:

Proposition 4.18. *For every $A \in (\text{salg})_{\mathbb{K}}$ we have*

$$G_V(A) = G_{\bar{0}}(A) G_1^{\leq}(A) \quad , \quad G_V(A) = G_1^{\leq}(A) G_{\bar{0}}(A)$$

We aim to show that the above decompositions are essentially unique. We need another lemma:

Lemma 4.19. *Let $A, B \in (\text{salg})_{\mathbb{K}}$, B a subsuperalgebra of A . Then $G_V(B)$ is a subgroup of $G_V(A)$.*

Proof. By definition G_V is a subgroup of $\text{GL}(V)$, so elements in $G_V(A)$ are realized as matrices with entries in A , and similarly for B replacing A . Then it is clear that any matrix in $G_V(B)$ is in $G_V(A)$, and two such matrices are equal in $G_V(B)$ if and only if they are equal in $G_V(A)$ too. \square

For the proof of the main result we need the following intermediate step:

Lemma 4.20. *Let $A \in (\text{salg})_{\mathbb{K}}$, and $g_{\pm}, f_{\pm} \in G_1^{\pm, <}(A)$. If $g_- g_+ = f_- f_+$, then $g_{\pm} = f_{\pm}$.*

Proof. To begin with, we write the element g_- as an ordered product $g_- = \prod_{d=1}^{N_-} (1 + \vartheta_d X_{\tilde{\gamma}_d})$, for some $\vartheta_d \in A_{\bar{1}}$, where the $\gamma_d \in \tilde{\Delta}_{\bar{1}}^-$ are all the negative odd roots, ordered as in Definition 4.15; also, hereafter $N_{\pm} = |\tilde{\Delta}_{\bar{1}}^{\pm}|$. Expanding the product on the right-hand side we get

$$g_- = \sum_{\substack{0 \leq k \leq N_- \\ 1 \leq d_1 < \dots < d_k \leq N_-}} (-1)^{\binom{k}{2}} \vartheta_{d_1} \dots \vartheta_{d_k} X_{\tilde{\gamma}_{d_1}} \dots X_{\tilde{\gamma}_{d_k}}$$

Similarly, $f_- = \prod_{b=1}^{N_-} (1 + \eta_b X_{\tilde{\gamma}_b})$, for some $\eta_b \in A_{\bar{1}}$, and then we have the expansion $f_-^{-1} = \prod_{b=N_-}^1 (1 + \eta_b X_{\tilde{\gamma}_b})^{-1} = \prod_{b=N_-}^1 (1 - \eta_b X_{\tilde{\gamma}_b}) = \sum_{\substack{0 \leq h \leq N_- \\ N_- \geq b_1 > \dots > b_h \geq 1}} (-1)^{\binom{h+1}{2}} \eta_{b_1} \dots \eta_{b_h} X_{\tilde{\gamma}_{b_1}} \dots X_{\tilde{\gamma}_{b_h}}.$

Now let $V = \bigoplus_{\mu} V_{\mu}$ be the splitting of V into a direct sum of weight spaces. Then $X_{\tilde{\eta}} \cdot V_{\mu} \subseteq V_{\mu+\tilde{\eta}}$ if $X_{\tilde{\eta}} \in \mathfrak{g}_{\tilde{\eta}}$ (for each root η and every weight μ): this and the previous expansions yield $(f_-^{-1} g_-) \cdot v_{\mu} \in \bigoplus_{\gamma^- \in \mathbb{N} \Delta_{\bar{1}}^-} V_{\mu+\gamma^-}$ for all weights μ and $v_{\mu} \in V_{\mu}(A)$, with $\mathbb{N} \Delta_{\bar{1}}^-$ being the \mathbb{N} -span of $\Delta_{\bar{1}}^-$. In a similar way — with parallel notation — we find also $(f_+ g_+^{-1}) \cdot v_{\mu} \in \bigoplus_{\gamma^+ \in \mathbb{N} \Delta_{\bar{1}}^+} V_{\mu+\gamma^+}.$

Now, the assumption $g_- g_+ = f_- f_+$ implies $f_-^{-1} g_- = f_+ g_+^{-1}$. Since $\mathbb{N} \Delta_{\bar{1}}^- \cap \mathbb{N} \Delta_{\bar{1}}^+ = \{0\}$, by the previous analysis the only weight space in which both $(f_-^{-1} g_-) \cdot v_{\mu}$ and $(f_+ g_+^{-1}) \cdot v_{\mu}$ may have a non-trivial weight component is $V_{\mu}(A)$ itself. In particular, letting $((f_-^{-1} g_-) \cdot v_{\mu})_{\mu+\gamma^-}$ be the weight component of $(f_-^{-1} g_-) \cdot v_{\mu}$ inside $V_{\mu+\gamma^-}(A)$, we have $((f_-^{-1} g_-) \cdot v_{\mu})_{\mu+\gamma^-} = 0$ for any $\gamma^- \in \mathbb{N} \Delta_{\bar{1}}^- \setminus \{0\}$. We shall now describe these components, and deduce that $g_- = f_-$.

From now on we use short-hand notation $\underline{\eta}_{\underline{b}} := \eta_{b_1} \dots \eta_{b_h}$, $\underline{X}_{\tilde{\gamma}_{\underline{b}}} := X_{\tilde{\gamma}_{b_1}} \dots X_{\tilde{\gamma}_{b_h}}$, and $\underline{\vartheta}_{\underline{d}} := \vartheta_{d_1} \dots \vartheta_{d_k}$, $\underline{X}_{\tilde{\gamma}_{\underline{d}}} := X_{\tilde{\gamma}_{d_1}} \dots X_{\tilde{\gamma}_{d_k}}$, for all ordered strings $\underline{b} := (b_1 > \dots > b_h)$ and $\underline{d} := (d_1 > \dots > d_k)$. By $|\underline{b}| := h$ and $|\underline{d}| := k$ we denote the *length* of \underline{b} and \underline{d} respectively. Now, we have

$$\begin{aligned} f_-^{-1} g_- &= \sum_{h,k=0}^{N_-} \sum_{\substack{|\underline{b}|=h \\ |\underline{d}|=k}} (-1)^{\binom{h+1}{2} + \binom{k}{2}} \underline{\eta}_{\underline{b}} \underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{\vartheta}_{\underline{d}} \underline{X}_{\tilde{\gamma}_{\underline{d}}} \\ &= \sum_{h,k=0}^{N_-} \sum_{\substack{|\underline{b}|=h \\ |\underline{d}|=k}} (-1)^{\binom{h+1}{2} + \binom{k}{2} + hk} \underline{\eta}_{\underline{b}} \underline{\vartheta}_{\underline{d}} \underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}} \end{aligned}$$

by the above expansions of f_-^{-1} and g_- . For every $\gamma^- \in \mathbb{N} \Delta_{\bar{1}}^-$, this last formula yields

$$((f_-^{-1} g_-) \cdot v_{\mu})_{\mu+\gamma^-} = \sum_{h,k=0}^{N_-} \sum_{\substack{|\underline{b}|=h, |\underline{d}|=k \\ \pi(\tilde{\gamma}_{\underline{b}}) + \pi(\tilde{\gamma}_{\underline{d}}) = \gamma^-}} (-1)^{\binom{h+1}{2} + \binom{k}{2} + hk} \underline{\eta}_{\underline{b}} \underline{\vartheta}_{\underline{d}} \underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}} \cdot v_{\mu}$$

where $\pi(\tilde{\gamma}_{\underline{b}}) := \sum_{i=1}^{|\underline{b}|} \pi(\tilde{\gamma}_{b_i})$ and $\pi(\tilde{\gamma}_{\underline{d}}) := \sum_{j=1}^{|\underline{d}|} \pi(\tilde{\gamma}_{d_j})$ — notation of §2.18. In particular, for a root $\gamma^- := \gamma_q^- \in \Delta_{\bar{1}}^-$ we can single out the only two summands in the last formula indexed by a pair of strings whose lengths are one and zero: then the whole formula reads

$$\begin{aligned} ((f_-^{-1} g_-) \cdot v_{\mu})_{\mu+\gamma_q^-} &= \sum_{p=1}^{N_-} (\vartheta_p - \eta_p) X_{\tilde{\gamma}_p} \cdot v_{\mu} + \\ &+ \sum_{\substack{h,k=0 \\ (h,k) \neq (0,1) \\ (h,k) \neq (1,0)}}^{N_-} \sum_{\substack{|\underline{b}|=h, |\underline{d}|=k \\ \pi(\tilde{\gamma}_{\underline{b}}) + \pi(\tilde{\gamma}_{\underline{d}}) = \gamma_q^-}} (-1)^{\binom{h+1}{2} + \binom{k}{2} + hk} \underline{\eta}_{\underline{b}} \underline{\vartheta}_{\underline{d}} \underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}} \cdot v_{\mu} \end{aligned} \quad (4.7)$$

For every $\gamma^- \in \Delta_1^-$, we call n -height of γ the highest number $|\gamma^-|$ such that γ^- itself is the sum of exactly $|\gamma|$ negative roots. Looking at (4.7), we see that all roots $\pi(\tilde{\gamma}_{b_i})$ or $\pi(\tilde{\gamma}_{d_j})$ involved in the strings \underline{b} or \underline{d} occurring in the last, double sum necessarily satisfy $|\pi(\tilde{\gamma}_{b_i})| \leq |\gamma_q^-|$, $|\pi(\tilde{\gamma}_{d_j})| \leq |\gamma_q^-|$. Now fix any $\gamma_q^- \in \Delta_1^-$ such that $|\gamma_q^-| = 1$: then our last remark implies that (4.7) reduces to $((f^{-1}g_-) \cdot v_\mu)_{\mu+\gamma_q^-} = \sum_{\substack{p=1 \\ \pi(\tilde{\gamma}_p) = \gamma_q^-}}^{N_-} (\vartheta_p - \eta_p) X_{\tilde{\gamma}_p} \cdot v_\mu$; thus from $((f^{-1}g_-) \cdot v_\mu)_{\mu+\gamma_q^-} = 0$ we eventually get $\sum_{\substack{p=1 \\ \pi(\tilde{\gamma}_p) = \gamma_q^-}}^{N_-} (\vartheta_p - \eta_p) X_{\tilde{\gamma}_p} \cdot v_\mu = 0$. As \mathfrak{g} acts faithfully on V , hence \mathfrak{g}_A does the same on V_A , we get $\sum_{\substack{p=1 \\ \pi(\tilde{\gamma}_p) = \gamma_q^-}}^{N_-} (\vartheta_p - \eta_p) X_{\tilde{\gamma}_p} = 0$ inside \mathfrak{g}_A . Since the $X_{\tilde{\gamma}_p}$ are part of a (Chevalley) basis of \mathfrak{g} , we conclude that $\vartheta_p = \eta_p$ for all $p \in \{1, \dots, N_-\}$ such that $\pi(\tilde{\gamma}_p)$ has n -height 1.

We shall now extend this result to all root vectors $X_{\tilde{\gamma}_p}$, by induction on the n -height of $\pi(\tilde{\gamma}_p^-)$.

Take in general any $\gamma_q^- \in \Delta_1^-$ with $|\gamma_q^-| > 1$: as induction hypothesis, we assume that for all $\tilde{\gamma}_{p'} \in \tilde{\Delta}_1^-$ such that $|\pi(\tilde{\gamma}_{p'})| \leq |\gamma_q^-|$ we have $\vartheta_{p'} = \eta_{p'}$. Consider the last, double sum in (4.7): any “monomial” in the root vectors occurring there is of the form

$$\underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}} = X_{\tilde{\gamma}_{b_1}} \cdots X_{\tilde{\gamma}_{b_h}} X_{\tilde{\gamma}_{d_1}} \cdots X_{\tilde{\gamma}_{d_k}} \quad (4.8)$$

with $b_1 > \dots > b_h$ and $d_1 < \dots < d_k$. Moreover, by construction we can assume also that $b_i \neq d_j$ for all i and j : indeed, if $b_i = d_j$ in some \underline{b}' and \underline{d}' , then the inductive assumption gives $\eta_{b_i} = \vartheta_{d_j}$, hence $\underline{\eta}_{\underline{b}'} \underline{\vartheta}_{\underline{d}'} = 0$ and so $(-1)^{\binom{h+1}{2} + \binom{k}{2} + hk} \underline{\eta}_{\underline{b}'} \underline{\vartheta}_{\underline{d}'} \underline{X}_{\tilde{\gamma}_{\underline{b}'}} \underline{X}_{\tilde{\gamma}_{\underline{d}'}} \cdot v_\mu = 0$.

Now, the monomial in (4.8) will occur a second time in the same sum as follows. A first case is when $b_h > d_1$: then $\underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}} = \underline{X}_{\tilde{\gamma}_{\underline{b}'}} \underline{X}_{\tilde{\gamma}_{\underline{d}'}}$ for $\underline{b}' := (b_1, \dots, b_h, d_1)$ and $\underline{d}' := (d_2, \dots, d_k)$; this includes also the case $h = 0$. The second case is $b_h < d_1$: then $\underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}} = \underline{X}_{\tilde{\gamma}_{\underline{b}'}} \underline{X}_{\tilde{\gamma}_{\underline{d}'}}$ for $\underline{b}' := (b_1, \dots, b_{h-1})$ and $\underline{d}' := (b_h, d_1, \dots, d_k)$; this makes sense for $k = 0$ as well.

In both cases, the new strings \underline{b}' and \underline{d}' enjoy the property $|\underline{b}'| = |\underline{b}| \pm 1$ and $|\underline{d}'| = |\underline{d}| \mp 1$.

Now, whenever we consider any such pair of monomials $\underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}}$ and $\underline{X}_{\tilde{\gamma}_{\underline{b}'}} \underline{X}_{\tilde{\gamma}_{\underline{d}'}}$ occurring in the last (double) sum in (4.7) and such that $\underline{X}_{\tilde{\gamma}_{\underline{d}}} \underline{X}_{\tilde{\gamma}_{\underline{d}'}} = \underline{X}_{\tilde{\gamma}_{\underline{b}'}} \underline{X}_{\tilde{\gamma}_{\underline{d}'}}$, by induction we have $\underline{\eta}_{\underline{b}} \underline{\vartheta}_{\underline{d}} = \underline{\eta}_{\underline{b}'} \underline{\vartheta}_{\underline{d}'}$. Even more, a direct check shows that for the signs involved one has $(-1)^{\binom{h+1}{2} + \binom{k}{2} + hk} + (-1)^{\binom{h'+1}{2} + \binom{k'}{2} + h'k'} = 0$ where $h = |\underline{b}|$, $k = |\underline{d}|$, $h' = |\underline{b}'|$, $k' = |\underline{d}'|$. Thus the two (identical!) monomials $\underline{X}_{\tilde{\gamma}_{\underline{b}}} \underline{X}_{\tilde{\gamma}_{\underline{d}}}$ and $\underline{X}_{\tilde{\gamma}_{\underline{b}'}} \underline{X}_{\tilde{\gamma}_{\underline{d}'}}$ in that sum cancel out each other.

The outcome is that the last, double sum in (4.7) is actually zero: thus (4.7) itself reduces to $((f^{-1}g_-) \cdot v_\mu)_{\mu+\gamma_q^-} = \sum_{\substack{p=1 \\ \pi(\tilde{\gamma}_p) = \gamma_q^-}}^{N_-} (\vartheta_p - \eta_p) X_{\tilde{\gamma}_p} \cdot v_\mu$: and, as before, one deduces $\vartheta_p = \eta_p$ for all p .

Thus the above induction argument yields $\vartheta_p = \eta_p$ for all $p = 1, \dots, N_-$, hence $g_- = f_-$.

An entirely similar analysis shows that $g_+ = f_+$, whence the claim is proved. \square

At last, we are ready for our main result:

Theorem 4.21. *For any $A \in (\text{salg})_{\mathbb{k}}$, the group product yields a bijection*

$$G_0(A) \times G_1^{-, <} (A) \times G_1^{+, <} (A) \longleftrightarrow G_V(A)$$

and all the similar bijections obtained by permuting the factors $G_1^{\pm, <} (A)$ and/or switching the factor $G_0(A)$ to the right.

Proof. We shall prove the first mentioned bijection.

In general, Proposition 4.18 gives $G_V(A) = G_0(A) G_1^{<} (A)$, so the product map from $G_0(A) \times G_1^{<} (A)$ to $G_V(A)$ is onto. But in particular, we can choose an ordering on Δ_1^- such that $\Delta_1^- \preceq \Delta_1^+$, hence $G_1^{<} (A) = G_1^{-, <} (A) G_1^{+, <} (A)$, so we are done for surjectivity.

To prove that the product map is injective amounts to showing that for any $g \in G_V(A)$ the factorization $g = g_0 g_- g_+$ with $g_0 \in G_0(A)$, $g_{\pm} \in G_1^{\pm, <} (A)$, is unique. In other words, if $g = g_0 g_- g_+ = f_0 f_- f_+$ with $g_0, f_0 \in G_0(A)$, $g_{\pm}, f_{\pm} \in G_1^{\pm, <} (A)$, we must prove $g_0 = f_0$, $g_{\pm} = f_{\pm}$.

By definition of $G_{\bar{0}}(A)$, both $g_{\bar{0}}$ and $f_{\bar{0}}$ are products of finitely many factors of type $x_{\bar{\alpha}}(t_{\bar{\alpha}})$ and $h_i(s_i)$ for some $t_{\bar{\alpha}} \in A_{\bar{0}}$, $s_i \in U(A_{\bar{0}})$ — with $\bar{\alpha} \in \tilde{\Delta}_{\bar{0}}$, $i = 1, \dots, r$. Moreover, there exist product expansions $g_{\pm} = \prod_{d=1}^{N_{\pm}} (1 + \vartheta_d^{\pm} X_{\tilde{\gamma}_d^{\pm}})$, $f_{\pm} = \prod_{d=1}^{N_{\pm}} (1 + \eta_d^{\pm} X_{\tilde{\gamma}_d^{\pm}})$ like in the proof of Lemma 4.20. We call B the superalgebra of A generated by all the $t_{\bar{\alpha}}$, the s_i , the ϑ_d^{\pm} and the η_d^{\pm} : this is finitely generated (as a superalgebra), and $B_{\bar{1}}$ is finitely generated as a $B_{\bar{0}}$ -module.

By Lemma 4.19, $G_V(B)$ is a subgroup into $G_V(A)$; therefore the identity $g_{\bar{0}}g_{-}g_{+} = f_{\bar{0}}f_{-}f_{+}$ holds inside $G_V(B)$ as well. Thus we can switch from A to B , i.e. we can assume from scratch that $A = B$. In particular A is finitely generated, so $A_{\bar{1}}$ is finitely generated as an $A_{\bar{0}}$ -module.

Consider in A the ideal $A_{\bar{1}}$, the submodules $A_{\bar{1}}^m$ (cf. section 2.1) and the ideal $(A_{\bar{1}}^m)$ of A generated by $A_{\bar{1}}^m$ ($m \in \mathbb{N}$): as $A_{\bar{1}}^m$ is *homogeneous*, we have $A/(A_{\bar{1}}^m) \in (\text{salg})_{\mathbb{k}}$. Moreover, as $A_{\bar{1}}$ is finitely generated (over $A_{\bar{0}}$), we have $A_{\bar{1}}^m = \{0\} = (A_{\bar{1}}^m)$ for $m \gg 0$. So it is enough to prove

$$g_{\bar{0}} \equiv f_{\bar{0}} \pmod{(A_{\bar{1}}^m)}, \quad g_{\pm} \equiv f_{\pm} \pmod{(A_{\bar{1}}^m)} \quad \forall \quad m \in \mathbb{N} \quad (4.9)$$

hereafter, for any $A' \in (\text{salg})_{\mathbb{k}}$, any I ideal of A' with $\pi_I : A' \twoheadrightarrow A'/I$ the canonical projection, by $x \equiv y \pmod{I}$ we mean that x and y in $G_V(A')$ have the same image in $G_V(A'/I)$ via $G_V(\pi_I)$.

We prove (4.9) by induction, the case $m = 0$ being clear, as there is no odd part.

Let (4.9) be true for even m . In particular, $g_{\pm} \equiv f_{\pm} \pmod{(A_{\bar{1}}^m)}$: then the proof of Lemma 4.20 applied to $G_V(A/A_{\bar{1}}^m)$ gives $\vartheta_d^{\pm} \equiv \eta_d^{\pm} \pmod{(A_{\bar{1}}^m)}$ for all d , hence $(\vartheta_d^{\pm} - \eta_d^{\pm}) \in (A_{\bar{1}}^m) \cap A_{\bar{1}} \subseteq (A_{\bar{1}}^{m+1})$, for all d , by an obvious parity argument. Thus $g_{\pm} \equiv f_{\pm} \pmod{(A_{\bar{1}}^{m+1})}$ too, hence from $g_{\bar{0}}g_{-}g_{+} = f_{\bar{0}}f_{-}f_{+}$ we get $g_{\bar{0}} \equiv f_{\bar{0}} \pmod{(A_{\bar{1}}^{m+1})}$ as well, that is (4.9) holds for $m+1$.

Let now (4.9) hold for odd m . Then $g_{\bar{0}} \equiv f_{\bar{0}} \pmod{(A_{\bar{1}}^m)}$; but $g_{\bar{0}}, f_{\bar{0}} \in G_{\bar{0}}(A) = G_{\bar{0}}(A_{\bar{0}})$ by definition, hence $g_{\bar{0}} \equiv f_{\bar{0}} \pmod{(A_{\bar{1}}^m) \cap A_{\bar{0}}}$. Therefore $g_{\bar{0}} \equiv f_{\bar{0}} \pmod{(A_{\bar{1}}^{m+1})}$ because, by an obvious parity argument again, one has $(A_{\bar{1}}^m) \cap A_{\bar{0}} \subseteq (A_{\bar{1}}^{m+1})$. Thus from $g_{\bar{0}}g_{-}g_{+} = f_{\bar{0}}f_{-}f_{+}$ we get also $g_{-}g_{+} \equiv f_{-}f_{+} \pmod{(A_{\bar{1}}^{m+1})}$. Then Lemma 4.20 again — now applied to $G(A/A_{\bar{1}}^{m+1})$ — eventually gives $g_{\pm} \equiv f_{\pm} \pmod{(A_{\bar{1}}^{m+1})}$, so that (4.9) holds for $m+1$ too. \square

The “overall consequence” of the last result is the following, straightforward corollary:

Corollary 4.22.

(a) *The group product yields functor isomorphisms*

$$G_{\bar{0}} \times G_{\bar{1}}^{-, <} \times G_{\bar{1}}^{+, <} \xrightarrow{\cong} G_V, \quad \mathbf{G}_{\bar{0}} \times \mathbf{G}_{\bar{1}}^{-, <} \times \mathbf{G}_{\bar{1}}^{+, <} \xrightarrow{\cong} \mathbf{G}_V$$

as well as those obtained by permuting the $(-)$ -factor and the $(+)$ -factor and/or moving the $(\bar{0})$ -factor to the right. All these induce similar functor isomorphisms with the left-hand side obtained by permuting the factors above, like $G_{\bar{1}}^{+, <} \times G_{\bar{0}} \times G_{\bar{1}}^{-, <} \xrightarrow{\cong} G_V$, $\mathbf{G}_{\bar{1}}^{-, <} \times \mathbf{G}_{\bar{0}} \times \mathbf{G}_{\bar{1}}^{+, <} \xrightarrow{\cong} \mathbf{G}_V$, etc.

(b) *The group product yields functor isomorphisms*

$$G_{\bar{0}}^{\pm} \times G_{\bar{1}}^{\pm, <} \xrightarrow{\cong} G_V^{\pm}, \quad \mathbf{G}_{\bar{0}}^{\pm} \times \mathbf{G}_{\bar{1}}^{\pm, <} \xrightarrow{\cong} \mathbf{G}_V^{\pm}, \quad G_{\bar{1}}^{\pm, <} \times G_{\bar{0}}^{\pm} \xrightarrow{\cong} G_V^{\pm}, \quad \mathbf{G}_{\bar{1}}^{\pm, <} \times \mathbf{G}_{\bar{0}}^{\pm} \xrightarrow{\cong} \mathbf{G}_V^{\pm}$$

(c) *Let \preceq be a total order on $\tilde{\Delta}_{\bar{1}}$ such that $\tilde{\Delta}_{\bar{1}}^{-} \preceq \tilde{\Delta}_{\bar{1}}^{+}$ or $\tilde{\Delta}_{\bar{1}}^{+} \preceq \tilde{\Delta}_{\bar{1}}^{-}$. Then the group product yields isomorphisms $G_{\bar{0}} \times G_{\bar{1}}^{\preceq} \xrightarrow{\cong} G_V$, $\mathbf{G}_{\bar{0}} \times \mathbf{G}_{\bar{1}}^{\preceq} \xrightarrow{\cong} \mathbf{G}_V$, $G_{\bar{1}}^{\preceq} \times G_{\bar{0}} \xrightarrow{\cong} G_V$, $\mathbf{G}_{\bar{1}}^{\preceq} \times \mathbf{G}_{\bar{0}} \xrightarrow{\cong} \mathbf{G}_V$.*

Yet another crucial step we can move on now is the following:

Proposition 4.23. *The functors $G_{\bar{1}}^{\pm, <} : (\text{salg})_{\mathbb{k}} \rightarrow (\text{sets})$ are representable: namely, they are the functor of points of the superscheme $\mathbb{A}_{\mathbb{k}}^{0|N_{\pm}}$, where $N_{\pm} := |\tilde{\Delta}_{\bar{1}}^{\pm}|$. In particular they are sheaves, hence $G_{\bar{1}}^{\pm, <} = \mathbf{G}_{\bar{1}}^{\pm, <}$. Similarly, for any total order in $\tilde{\Delta}_{\bar{1}}$ such that $\tilde{\Delta}_{\bar{1}}^{-} \preceq \tilde{\Delta}_{\bar{1}}^{+}$ or $\tilde{\Delta}_{\bar{1}}^{+} \preceq \tilde{\Delta}_{\bar{1}}^{-}$, we have $G_{\bar{1}}^{\preceq} = \mathbf{G}_{\bar{1}}^{\preceq} \cong \mathbb{A}^{0|N}$ as super-schemes, with $N := |\tilde{\Delta}_{\bar{1}}| = N_{+} + N_{-}$.*

Proof. Clearly, there exists a natural transformation $\Psi^{\pm} : \mathbb{A}_{\mathbb{k}}^{0|N_{\pm}} \rightarrow G_{\bar{1}}^{\pm, <}$ given on objects by

$$\Psi^{\pm}(A) : \mathbb{A}_{\mathbb{k}}^{0|N_{\pm}}(A) \rightarrow G_{\bar{1}}^{\pm, <}(A), \quad (\vartheta_1, \dots, \vartheta_{N_{\pm}}) \mapsto \prod_{i=1}^{N_{\pm}} x_{\tilde{\gamma}_i}(\vartheta_i)$$

Now given $g_1^\pm = \prod_{i=1}^{N_\pm} x_{\tilde{\gamma}_i}(\vartheta'_i) \in G_1^{\pm, <}(A)$, $h_1^\pm = \prod_{i=1}^{N_\pm} x_{\tilde{\gamma}_i}(\vartheta''_i) \in G_1^{\pm, <}(A)$, assume that $g_1^\pm = h_1^\pm$, hence $h_1^\pm (g_1^\pm)^{-1} = 1$. Then we get $(\vartheta'_1, \dots, \vartheta'_{N_\pm}) = (\vartheta''_1, \dots, \vartheta''_{N_\pm})$ just as showed in the proof of Lemma 4.20. This means that Ψ^\pm is an isomorphism of functors, which proves the first part of the claim. The last part of the claim then follows like for Corollary 4.22(c). \square

Finally, we prove that the supergroup functors \mathbf{G}_V are affine algebraic:

Theorem 4.24. *Every functor \mathbf{G}_V is an affine algebraic supergroup.*

Proof. First, $\mathbf{G}_1^{-, <}$ and $\mathbf{G}_1^{+, <}$ are affine, and algebraic, by Proposition 4.23; moreover, by Proposition 4.10(c), or by Proposition 4.12, \mathbf{G}_0 is affine algebraic as well. Now Corollary 4.22 gives $\mathbf{G}_V \cong \mathbf{G}_0 \times \mathbf{G}_1^{-, <} \times \mathbf{G}_1^{+, <}$ as superschemes. As any direct product of affine algebraic superschemes is affine algebraic too (see [3], Ch. 10), we can eventually conclude the same for \mathbf{G}_V . \square

Remark 4.25. Theorem 4.24 and Proposition 4.23 together show that $\mathbf{G}_V \cong \mathbf{G}_0 \times \mathbf{G}_1^{-, <} \times \mathbf{G}_1^{+, <}$ as superschemes. As $\mathbf{G}_1^{\pm, <} \cong G_1^{\pm, <}$ is generated by the one-parameter supersubgroups $x_{\tilde{\gamma}} (\tilde{\gamma} \in \tilde{\Delta}_1^\pm)$, we conclude that \mathbf{G}_V can also be described as $\mathbf{G}_V(A) = \langle \mathbf{G}_0(A) \cup \{x_{\tilde{\gamma}}(A)\}_{\tilde{\gamma} \in \tilde{\Delta}_1} \rangle$ for all $A \in (\text{salg})_{\mathbb{k}}$. Even more, as $\mathbf{G}_0 \cong \mathbf{G}_0 \times \mathbf{G}_{0\uparrow}$ by Proposition 4.12 and $\mathbf{G}_{0\uparrow} \cong \times_{\tilde{\alpha} \in \tilde{\Delta}_{0\uparrow}} x_{\tilde{\alpha}}$ by Proposition 4.14, we have also $\mathbf{G}_V(A) = \langle \mathbf{G}_0(A) \cup \{x_{\tilde{\alpha}}(A)\}_{\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}_0} \rangle$, for all $A \in (\text{salg})_{\mathbb{k}}$.

We finish with an additional, non-obvious remark: under mild assumptions, the supergroup \mathbf{G}_V , which by construction is a supersubgroup of $\text{GL}(V)$, is indeed a *closed* one:

Proposition 4.26. *Assume that \mathfrak{g}_1 as a \mathbb{k} -submodule of $\mathfrak{gl}(V)_1$ is a direct summand. Then \mathbf{G}_V is a closed supersubgroup of $\text{GL}(V)$. In particular, this is always true if \mathbb{k} is a field.*

Proof. By construction we have that $\mathbf{G} \leq \text{GL}(V)$. Consider the factorization $\mathbf{G}_V = \mathbf{G}_0 \times \mathbf{G}_1^{<}$ in Corollary 4.22(c): by construction, \mathbf{G}_0 is just a classical algebraic group(-scheme), embedded into $\text{GL}(V)$ as a closed subgroup, therefore it is enough to show that $\mathbf{G}_1^{<}$ is closed too.

Recall that $\text{GL}(V)$ can be realized as an open supersubscheme of $\text{End}(V) = \text{Mat}_{m|n}(\mathbb{k})$, where $m|n$ is the (super)rank of V ; so it is enough to prove that $\mathbf{G}_1^{<}$ is closed in $\text{Mat}_{m|n}(\mathbb{k})$; recall also

$$\mathcal{O}(\text{End}(V)) = \mathcal{O}(\text{Mat}_{m|n}(\mathbb{k})) = \mathbb{k} \left[\{x'_{i,j}, x''_{r,s}, \xi'_{i,s}, \xi''_{r,j}\}_{i,j=1,\dots,m; r,s=1,\dots,n} \right]$$

Using Proposition 4.23, we identify $\mathbb{A}_{\mathbb{k}}^{0|N} \cong \mathbf{G}_1^{<}$ so that the point 0 in $\mathbb{A}_{\mathbb{k}}^{0|N}$ corresponds to the identity I in $\mathbf{G}_1^{<}$. Then the tangent superspace to $\mathbf{G}_1^{<}$ at I corresponds to the tangent superspace to $\mathbb{A}_{\mathbb{k}}^{0|N}$ at 0, naturally identified with $\mathbb{A}_{\mathbb{k}}^{0|N}$ again. By the assumption on \mathfrak{g}_1 , we can complete the \mathbb{k} -basis $\{X_{\tilde{\gamma}_1}, \dots, X_{\tilde{\gamma}_N}\}$ of \mathfrak{g}_1 to a \mathbb{k} -basis of $\mathfrak{gl}(V)_1$: this in turn correspond to a “change of odd variables” in $\mathcal{O}(\text{End}(V))$, from $\{\xi'_{i,s}, \xi''_{r,j}\}_{i,j=1,\dots,m; r,s=1,\dots,n}$ to some new set of odd variables, say $\{\hat{\xi}_1, \dots, \hat{\xi}_{2mn}\}$, such that $\langle X_{\tilde{\gamma}_h}, \hat{\xi}_k \rangle = \delta_{h,k}$. Letting \mathcal{J} be the embedding map of $\mathbf{G}_1^{<}$ into $\text{End}(V)$, the tangent map $d_I \mathcal{J}$ (of \mathcal{J} at I) is expressed by a 2-by-2 block matrix whose only non-trivial block (in the right-bottom corner) is $\left(\frac{\partial \hat{\xi}_h}{\partial \vartheta_k} \right)_{h,k=1,\dots,N}$. Now, given $A \in (\text{salg})_{\mathbb{k}}$,

any $g = \prod_{i=1}^N x_{\tilde{\gamma}_i}(\vartheta_i) \in \mathbf{G}_1^{<}(A)$ expands as $g = \prod_{i=1}^N x_{\tilde{\gamma}_i}(\vartheta_i) = I + \sum_{i=1}^N \vartheta_i X_{\tilde{\gamma}_i} + \mathcal{O}(2)$, where $\mathcal{O}(2)$ stands for some element in $\mathfrak{gl}(V(A)) = A_0 \otimes_{\mathbb{k}} \mathfrak{gl}(V)_0 + A_1 \otimes_{\mathbb{k}} \mathfrak{gl}(V)_1$ whose (non-zero) coefficients in A_0 and A_1 actually belong to A_1^2 (cf. Subsec. 2.1). This implies that $\frac{\partial \hat{\xi}_h}{\partial \vartheta_k} = \delta_{h,k}$, so that the only non-trivial block in the matrix of $d_I \mathcal{J}$ is the identity matrix of size N . Thanks to this last remark, we can adapt the *Inverse Function Theorem* and its corollaries (see [3], §§5.1–2) to the present context: the outcome is that there exists “a change of variables”

$$\{x'_{i,j}, x''_{r,s}\}_{i,j=1,\dots,m; r,s=1,\dots,n} \mapsto \{\tilde{x}'_{i,j}, \tilde{x}''_{r,s}\}_{i,j=1,\dots,m; r,s=1,\dots,n}, \quad \{\hat{\xi}_t\}_{t=1,\dots,N} \mapsto \{\tilde{\xi}_t\}_{t=1,\dots,N};$$

such that the morphism of superalgebras $\mathcal{J}^* : \mathcal{O}(\text{End}(V)) \rightarrow \mathcal{O}(\mathbf{G}_1^{<})$ corresponding to \mathcal{J} is given by mapping $\tilde{x}'_{i,j} \mapsto \delta_{i,j}$, $\tilde{x}''_{r,s} \mapsto \delta_{r,s}$, $\tilde{\xi}_t \mapsto \vartheta_t$ for $t \leq N$, $\tilde{\xi}_t \mapsto 0$ for $t > N$. In turn, $\mathbf{G}_1^{<} = \text{Im}(\mathcal{J})$ is the zero locus $\text{Ker}(\mathcal{J}^*)$, hence it is a closed supersubscheme of $\text{End}(V)$. \square

4.5 The dependence on V

The construction of the supergroups \mathbf{G}_V was made via the Lie superalgebra \mathfrak{g} and the \mathfrak{g} -module V . So we have to clarify how supergroups attached to different \mathfrak{g} -modules are related among them. Moreover, the construction involves the choice of an admissible \mathbb{Z} -lattice M in V : nevertheless, we shall presently show that the outcome, i.e. \mathbf{G}_V itself, is actually independent of that choice.

4.27. The weight lattice of \mathbf{G}_V . Let L_w be the lattice of all “integral weights” of \mathfrak{g}_0 (in short, “the weight lattice of \mathfrak{g}_0 ”), using standard terminology, cf. for instance [13]: in particular, these are weights with respect to the Cartan subalgebra \mathfrak{h} of \mathfrak{g} . Also, we let L_r be the lattice spanned by all the \mathfrak{h} -roots of \mathfrak{g} (in short, “the root lattice”): here by “ \mathfrak{h} -roots” we mean the eigenvalues in \mathfrak{g} of the adjoint action of \mathfrak{h} (again), not of $\bar{\mathfrak{h}}$; these \mathfrak{h} -roots are just the restrictions (as linear functionals) from $\bar{\mathfrak{h}}$ to \mathfrak{h} of the roots of \mathfrak{g} considered in §2.18 (which might be called “ $\bar{\mathfrak{h}}$ -roots”). Actually, nothing changes in all cases but $H(n)$: for the latter, an explicit description of the \mathfrak{h} -roots follows from considering the description of the $\bar{\mathfrak{h}}$ -roots and reading it modulo δ .

From §2.18 we see that the root lattice L_r is spanned by the weights $\varepsilon_1, \dots, \varepsilon_r$ of the defining representation of the reductive Lie algebra \mathfrak{g}_0 . On the other hand, the weight lattice L_w is spanned by the so-called fundamental dominant weights $\omega_1, \dots, \omega_r$. Now, looking at the relationship between the ε_i and the ω_j one sees that the quotient module L_w/L_r is

- trivial, when \mathfrak{g} is of type W , S or \tilde{S} ,
- isomorphic to \mathbb{Z}_2 , when \mathfrak{g} is of type $H(2r+1)$,
- isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, when \mathfrak{g} is of type $H(2r)$;

therefore, in all cases L_w is just “slightly bigger” than L_r .

Now let \mathbf{G}_V be a supergroup constructed as in section 4, associated with the Lie superalgebra \mathfrak{g} of Cartan type and with a faithful, rational, finite dimensional \mathfrak{g} -module V with admissible lattice M . Now Corollary 4.22, Proposition 4.23, Proposition 4.12 and Proposition 4.14 altogether give

$$\mathbf{G}_V \cong \mathbf{G}_0 \times \mathbf{G}_1^< \cong \mathbf{G}_0 \times \mathbf{G}_{0\uparrow} \times \mathbf{G}_1^< \cong \mathbf{G}_0 \times \mathbb{A}^{N_{0\uparrow}}|^0 \times \mathbb{A}^{0|N} \cong \mathbf{G}_0 \times \mathbb{A}^{N_{0\uparrow}}|N$$

(with notations used there), i.e. $\mathbf{G}_V \cong \mathbf{G}_0 \times \mathbb{A}^{N_{0\uparrow}}|N$. By Proposition 4.9, $\mathbf{G}_0 \cong \mathbf{Ch}_V$ is a classical, split reductive algebraic group. By classical theory we know that $\mathbf{G}_0 \cong \mathbf{Ch}_V$ depends only on the lattice of \mathfrak{g}_0 -weights (= \mathfrak{g} -weights) of V : we denote this weight lattice by L_V .

Now, for the lattice L_V associated with the supergroup \mathbf{G}_V we have clearly $L_r \subseteq L_V \subseteq L_w$. By the remarks above about L_w/L_r , we have that L_V is always “very close” to L_r or L_w : in particular, we always have equalities $L_r = L_V = L_w$ when \mathfrak{g} is of type W , S or \tilde{S} (i.e., not H).

Let now \mathbf{G}_V and $\mathbf{G}'_{V'}$ be two Cartan supergroups obtained from \mathfrak{g} via different \mathfrak{g} -modules V and V' . We let $x_{\tilde{\alpha}}(\mathbf{t})$, $x'_{\tilde{\alpha}}(\mathbf{t})$, and $h_H(u)$, $h'_H(u)$, be the points of the one-parameter supersubgroups in \mathbf{G}_V and $\mathbf{G}'_{V'}$, associated with $\tilde{\alpha} \in \tilde{\Delta}$, $\mathbf{t} \in A_{\tilde{0}} \cup A_{\tilde{1}}$, and $H \in \mathfrak{h}_{\mathbb{Z}}$, $u \in U(A_{\tilde{0}})$ — cf. subsection 4.1.

Lemma 4.28. *Let $\phi : \mathbf{G}_V \rightarrow \mathbf{G}'_{V'}$ be a morphism of the supergroups mentioned above. Assume that $\phi_A(\mathbf{G}_0(A)) = \mathbf{G}'_0(A)$ and $\phi_A(x_{\tilde{\alpha}}(\mathbf{t})) = x'_{\tilde{\alpha}}(\mathbf{t})$ for all $A \in (\text{salg})_{\mathbb{k}}$, $\mathbf{t} \in A_{\tilde{0}} \cup A_{\tilde{1}}$, $\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}_0$.*

Then $\text{Ker}(\phi) \subseteq Z(\mathbf{G}_0)$, where $Z(\mathbf{G}_0)$ is the center of \mathbf{G}_0 .

Proof. First, note that $\tilde{\Delta} \setminus \tilde{\Delta}_0 = \tilde{\Delta}_{0\uparrow} \cup \tilde{\Delta}_{\tilde{1}}^- \cup \tilde{\Delta}_{\tilde{1}}^+$, and fix a total order \preceq on this set such that $\tilde{\Delta}_{0\uparrow} \preceq \tilde{\Delta}_{\tilde{1}}^- \preceq \tilde{\Delta}_{\tilde{1}}^+$. Then take $g \in \mathbf{G}_V(A)$ with $g \in \text{Ker}(\phi_A)$. By Corollary 4.22, Proposition 4.23, Proposition 4.12, Proposition 4.14 and Proposition 4.9, there is a unique factorization of g

$$g = g_0 \cdot \prod_{\tilde{\beta} \in \tilde{\Delta}_{0\uparrow}} x_{\tilde{\beta}}(t_{\tilde{\beta}}) \cdot \prod_{\tilde{\gamma} \in \tilde{\Delta}_{\tilde{1}}^-} x_{\tilde{\gamma}}(\vartheta_{\tilde{\gamma}}) \cdot \prod_{\tilde{\gamma} \in \tilde{\Delta}_{\tilde{1}}^+} x_{\tilde{\gamma}}(\vartheta_{\tilde{\gamma}}) \quad (4.10)$$

(all products being ordered with respect to \preceq) for some $g_0 \in \mathbf{G}_0(A)$, $t_{\tilde{\beta}} \in A_{\tilde{0}}$ and $\vartheta_{\tilde{\gamma}} \in A_{\tilde{1}}$. A similar factorization also holds for $\phi_A(g)$ in $\mathbf{G}'_{V'}(A)$. All this along with $\phi_A(g) = e_{\mathbf{G}'_{V'}(A)}$, with

$$\phi_A(g) = \phi_A(g_0) \cdot \prod_{\tilde{\beta} \in \tilde{\Delta}_{0\uparrow}} \phi_A(x_{\tilde{\beta}}(t_{\tilde{\beta}})) \cdot \prod_{\tilde{\gamma} \in \tilde{\Delta}_{\tilde{1}}^-} \phi_A(x_{\tilde{\gamma}}(\vartheta_{\tilde{\gamma}})) \cdot \prod_{\tilde{\gamma} \in \tilde{\Delta}_{\tilde{1}}^+} \phi_A(x_{\tilde{\gamma}}(\vartheta_{\tilde{\gamma}}))$$

and with the unicity of the factorization of $\phi_A(g)$ implies, by the assumption $\phi_A(x_{\tilde{\alpha}}(\mathbf{t})) = x'_{\tilde{\alpha}}(\mathbf{t})$, that all factors in the product on right-hand side here above are trivial. In turn, all factors but g_0 on right-hand side of (4.10) are trivial too; therefore $g = g_0 \in \mathbf{G}_0(A) \cap \text{Ker}(\phi_A) = \text{Ker}(\phi_A|_{\mathbf{G}_0(A)})$.

By assumption $\phi_A|_{\mathbf{G}_0} : \mathbf{G}_0 \rightarrow \mathbf{G}'_0$ is an epimorphism, with \mathbf{G}_0 and \mathbf{G}'_0 being connected split reductive algebraic groups over \mathbb{Z} having tangent Lie algebra \mathfrak{g} (by Proposition 4.9), so $d\phi : \mathbf{G}_0 \rightarrow \mathbf{G}'_0$ is an isomorphism. By classical theory this forces $\text{Ker}(\phi_A|_{\mathbf{G}_0}) \subseteq Z(\mathbf{G}_0)$. \square

Using this last result, we can now show that the relation between supergroups \mathbf{G}_V associated with different \mathfrak{g} -modules V is the same as in the classical setting. The result reads as follows:

Proposition 4.29. *Let \mathbf{G}_V and $\mathbf{G}'_{V'}$ be two affine supergroups constructed using two \mathfrak{g} -modules V and V' as in subsection 3.4. If $L_V \supseteq L_{V'}$, then there exists a unique morphism $\phi : \mathbf{G}_V \rightarrow \mathbf{G}'_{V'}$, such that $\text{Ker}(\phi) \subseteq Z(\mathbf{G}_0)$ and $\phi_A(x_{\tilde{\alpha}}(\mathbf{t})) = x'_{\tilde{\alpha}}(\mathbf{t})$ for every $A \in (\text{salg})_{\mathbb{K}}$, $\mathbf{t} \in A_{\tilde{0}} \cup A_{\tilde{1}}$, $\tilde{\alpha} \in \tilde{\Delta} \setminus \tilde{\Delta}_0$. Moreover, ϕ is an isomorphism if and only if $L_V = L_{V'}$.*

Proof. By classical theory, if $L_V \supseteq L_{V'}$ there exists a well-defined epimorphism $\phi_0 : \mathbf{G}_0 \rightarrow \mathbf{G}'_0$: in particular, we can (and we do) choose it so that $d\phi_0 = \text{id}_{\mathfrak{g}_0}$ (recall that $\text{Lie}(\mathbf{G}_0) = \mathfrak{g}_0 = \text{Lie}(\mathbf{G}'_0)$ by Proposition 4.9). As a consequence, we have that ϕ_0 acts on one-parameter (additive and multiplicative) subgroups of \mathbf{G}_0 as $\phi_0(x_{\tilde{\alpha}}(t)) = x'_{\tilde{\alpha}}(t)$ and $\phi_0(h_H(u)) = h'_H(u)$.

Now we extend ϕ_0 to a morphism $\phi : \mathbf{G}_V \rightarrow \mathbf{G}'_{V'}$ as follows. Fix $A \in (\text{salg})_{\mathbb{K}}$ and a total order in $\tilde{\Delta} \setminus \tilde{\Delta}_0$; use the unique factorization in $\mathbf{G}_V(A)$ — like in the proof of Lemma 4.28 — to factor g as in (4.10). Then define $\phi_A(g) := \phi_0(g_0) \cdot \prod_{\tilde{\beta} \in \tilde{\Delta}_{0+}} x'_{\tilde{\beta}}(t_{\tilde{\beta}}) \cdot \prod_{\tilde{\gamma}^- \in \tilde{\Delta}_{\tilde{1}}^-} x'_{\tilde{\gamma}^-}(\vartheta_{\tilde{\gamma}^-}) \cdot \prod_{\tilde{\gamma}^+ \in \tilde{\Delta}_{\tilde{1}}^+} x'_{\tilde{\gamma}^+}(\vartheta_{\tilde{\gamma}^+})$.

This gives a well-defined map $\phi_A : \mathbf{G}_V(A) \rightarrow \mathbf{G}'_{V'}(A)$, which by construction is functorial in A : thus we have a natural transformation ϕ — a morphism of superschemes — from \mathbf{G}_V to $\mathbf{G}'_{V'}$.

Moreover, this ϕ is also a morphism of supergroups. In fact, if $A \in (\text{salg})_{\mathbb{K}}$ is local then ϕ_A is a group morphism: indeed, $\mathbf{G}_V(A) = G_V(A)$ and $\mathbf{G}'_{V'}(A) = G'_{V'}(A)$, and we have $\phi_A(gh) = \phi_A(g)\phi_A(h)$ because all the relations used to commute elements in $G_V(A)$ or in $G'_{V'}(A)$ so to write a given element in “normal form” as in (4.10) actually *do not depend on the chosen representation*. Finally, by Proposition A.12 in [9], we have that ϕ is uniquely determined by its effect on local superalgebras, on which we saw it is a morphism: thus we conclude that ϕ is globally a morphism.

By construction ϕ is also onto. Thus all assumptions of Lemma 4.28 hold, and we can conclude that $\text{Ker}(\phi) \subseteq Z(\mathbf{G}_0)$ and $\phi_A(x_{\tilde{\alpha}}(\mathbf{t})) = x'_{\tilde{\alpha}}(\mathbf{t})$. Finally, again by construction ϕ is an isomorphism if and only if ϕ_0 is an isomorphism itself: but this in turn holds if and only if $L_V = L_{V'}$. \square

As a direct consequence, we have the following “independence result”:

Corollary 4.30. *Every supergroup \mathbf{G}_V constructed so far is independent, up to isomorphism, of the choice (which is needed in the very construction) of an admissible lattice M of V .*

Proof. Let M and M' be two admissible lattices of V , and set $V' := V$. Construct \mathbf{G}_V and $\mathbf{G}_{V'}$ using respectively M and M' : then we have $L_V = L_{V'}$, so Proposition 4.29 gives $\mathbf{G}_V \cong \mathbf{G}_{V'}$. \square

4.6 Lie’s Third Theorem for the supergroups \mathbf{G}_V

Let \mathbf{G}_V be an (affine) supergroup over the ring \mathbb{K} , built out of the Lie superalgebra \mathfrak{g} (of Cartan type) over \mathbb{K} and of the \mathfrak{g} -module V as in subsection 4.2. In subsection 4.1 we have introduced the Lie superalgebra $\mathfrak{g}_{V,\mathbb{K}} := \mathbb{K} \otimes_{\mathbb{Z}} \mathfrak{g}_V$ over \mathbb{K} starting from the \mathbb{Z} -lattice \mathfrak{g}_V . We now show that the algebraic supergroup \mathbf{G}_V has $\mathfrak{g}_{V,\mathbb{K}}$ as its tangent Lie superalgebra.

We start recalling how to associate a Lie superalgebra with a supergroup scheme ([3], §§11.2–5).

4.31. The Lie superalgebra of a supergroup scheme. For a given $A \in (\text{salg})_{\mathbb{K}}$ let $A[\epsilon] := A[x]/(x^2)$ be the superalgebra of dual numbers, in which $\epsilon := x \bmod (x^2)$ is taken to be *even*. Then $A[\epsilon] = A \oplus A\epsilon$, and there are natural morphisms $i : A \rightarrow A[\epsilon]$, $a \mapsto a$, and $p : A[\epsilon] \rightarrow A$, $(a + a'\epsilon) \mapsto a$, such that $p \circ i = \text{id}_A$. Given a supergroup \mathbb{K} -functor $G : (\text{salg})_{\mathbb{K}} \rightarrow (\text{groups})$,

denote the morphism associated with $p : A[\epsilon] \rightarrow A$ by $G(p)_A : G(A(\epsilon)) \rightarrow G(A)$. This gives a unique Lie algebra valued functor $Lie(G) : (\text{salg})_{\mathbb{k}} \rightarrow (\text{Lie})_{\mathbb{k}}$ given on objects by $Lie(G)(A) := \text{Ker}(G(p)_A)$. For the Lie structure, one first defines the *adjoint action* $Ad : G \rightarrow \text{GL}(Lie(G))$ of G on $Lie(G)$ as $Ad(g)(x) := G(i)(g) \cdot x \cdot (G(i)(g))^{-1}$ for all $g \in G(A)$, $x \in Lie(G)(A)$. Then one defines the *adjoint morphism* $ad := Lie(Ad) : Lie(G) \rightarrow Lie(\text{GL}(Lie(G))) := \text{End}(Lie(G))$, and finally sets $[x, y] := ad(x)(y)$ for all $x, y \in Lie(G)(A)$. Further details are in [3], §§11.3–5; note that the authors there assume \mathbb{k} to be a field, yet this is *not* required for the present context.

When G is (the functor of points of) a supergroup \mathbb{k} -scheme and \mathbb{k} is a field, the functor $Lie(G)$ is quasi-representable (cf. §2.9): indeed, it can be identified with (the functor of points of) the tangent superspace at the identity of G , denoted $T_e(G)$. In turn, $T_e(G)$ bears a structure of Lie \mathbb{k} -superalgebra, as usual (cf. [3], §11.4.); moreover, we point out that it bears also a canonical 2-operation, which can be given using the (standard) identification of $T_e(G)$ with the \mathbb{k} -superalgebra of left-invariant superderivations (into itself) of $\mathcal{O}(G)$, the Hopf \mathbb{k} -superalgebra representing G .

We shall presently see that for Cartan \mathbb{k} -supergroups \mathbf{G}_V this is the case also if \mathbb{k} is not a field: we shall then denote by $Lie(G)$ both the above functor and the associated \mathbb{k} -supermodule. Note that this is also the case for the \mathbb{k} -supergroup $Lie(\text{GL}_{m|n})$: indeed, it is well known that, whatever \mathbb{k} is, the functor $Lie(\text{GL}_{m|n})$ is quasi-representable, and identifies with the Lie \mathbb{k} -superalgebra $\mathfrak{gl}_{m|n}$; as the latter is free (as a \mathbb{k} -module) of finite rank, $Lie(\text{GL}_{m|n})$ is in fact representable too.

Eventually, we are now ready for the main result of this subsection.

Theorem 4.32. *Let \mathbf{G}_V be the affine supergroup of Cartan type built upon \mathfrak{g} and the \mathfrak{g} -module V (cf. section 4.2). Then $Lie(\mathbf{G}_V)$ is quasi-representable, and actually representable, namely $Lie(\mathbf{G}_V) = \mathcal{L}_{\mathfrak{g}_{V,\mathbb{k}}}$ as functors from $(\text{salg})_{\mathbb{k}}$ to $(\text{Lie})_{\mathbb{k}}$.*

Proof. The result follows from sheer computations: as everything takes place inside $\text{GL}(V)$, one can argue like in the standard example of $Lie(\text{GL}_{m|n})$ — which can be found, e.g., in [3], §11.3.

First, from the decomposition $\mathbf{G}_V = \mathbf{G}_{\bar{0}} \times \mathbf{G}_{\bar{1}}^< = \mathbf{G}_{\bar{0}} \times (\times_{\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}} x_{\tilde{\gamma}})$ — see Corollary 4.22 and Proposition 4.23 — we find at once that

$$\begin{aligned} Lie(\mathbf{G}_V)(A) &= Lie(\mathbf{G}_{\bar{0}})(A_{\bar{0}}) \times (\times_{\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}} (1 + \epsilon A_{\bar{1}} X_{\tilde{\gamma}})) = \\ &= Lie(\mathbf{G}_{\bar{0}})(A_{\bar{0}}) \times (1 + \epsilon \sum_{\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}} A_{\bar{1}} X_{\tilde{\gamma}}) = Lie(\mathbf{G}_{\bar{0}})(A_{\bar{0}}) \times (1 + \epsilon A_{\bar{1}} \otimes_{\mathbb{k}} \mathfrak{g}_{\bar{1}}) \end{aligned}$$

Second, by the results in subsection 4.3 and by the classical theory of Chevalley groups we know that $Lie(\mathbf{G}_{\bar{0}})$ is quasi-representable (and actually representable), with $Lie(\mathbf{G}_{\bar{0}}) = \mathcal{L}_{(\mathfrak{g}_{\bar{0}})_{V,\mathbb{k}}}$ where $(\mathfrak{g}_{\bar{0}})_{V,\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}} (\mathfrak{g}_{\bar{0}})_V$ and $(\mathfrak{g}_{\bar{0}})_V := \{X \in \mathfrak{g}_{\bar{0}} \mid X.M \subseteq M\} = \mathfrak{h}_V \oplus (\bigoplus_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}}} \mathbb{Z} X_{\tilde{\alpha}})$ much like in Proposition 3.17 and in subsection 4.1. From this and the previous remark, it follows that

$$Lie(\mathbf{G}_V)(A) = \mathcal{L}_{(\mathfrak{g}_{\bar{0}})_{V,\mathbb{k}}}(A_{\bar{0}}) \times (1 + \epsilon A_{\bar{1}} \otimes_{\mathbb{k}} \mathfrak{g}_{\bar{1}}) = A_{\bar{0}} \otimes_{\mathbb{k}} (\mathfrak{g}_{\bar{0}})_{V,\mathbb{k}} + A_{\bar{1}} \otimes_{\mathbb{k}} \mathfrak{g}_{\bar{1}} = \mathcal{L}_{(\mathfrak{g}_{\bar{0}})_{V,\mathbb{k}} \oplus \mathfrak{g}_{\bar{1}}}(A)$$

so that $Lie(\mathbf{G}_V) = \mathcal{L}_{\mathfrak{g}_{V,\mathbb{k}}}$ — as claimed — because $(\mathfrak{g}_{\bar{0}})_{V,\mathbb{k}} \oplus \mathfrak{g}_{\bar{1}} = \mathfrak{g}_{V,\mathbb{k}}$. In particular, as $\mathfrak{g}_{V,\mathbb{k}}$ is free of finite rank it follows that $Lie(\mathbf{G}_V)$ is representable too. \square

4.7 Special supersubgroups of \mathbf{G}_V

In subsections 4.2 and 4.3 we considered the (super)subgroups $\mathbf{G}_{\bar{0}}$ and $\mathbf{G}_{\bar{0}}$ of \mathbf{G}_V . We introduce now some other remarkable supersubgroups, associated with special Lie supersubalgebras of \mathfrak{g} .

Definition 4.33. Fix a splitting $\tilde{\Delta}_0 = \tilde{\Delta}_0^+ \amalg \tilde{\Delta}_0^-$ of the classical root system $\tilde{\Delta}_0 = \Delta_0$ (of \mathfrak{g}_0) into positive and negative roots. For all $A \in (\text{salg})_{\mathbb{k}}$ and $t > -1$, define the subgroups of $G_V(A)$

$$\begin{aligned} G_{-1\uparrow}(A) &:= \left\langle h_H(A), x_{\tilde{\alpha}}(A) \mid H \in \mathfrak{h}_{\mathbb{Z}}, \tilde{\alpha} \in \amalg_{z > -1} \tilde{\Delta}_z \right\rangle, & (G_{-1\uparrow})_{\bar{0}}(A) &:= G_{-1\uparrow}(A) \cap G_{\bar{0}}(A) \\ G_{t\uparrow}(A) &:= \left\langle x_{\tilde{\alpha}}(A) \mid \tilde{\alpha} \in \amalg_{z > t} \tilde{\Delta}_z \right\rangle, & (G_{t\uparrow})_{\bar{0}}(A) &:= G_{t\uparrow}(A) \cap G_{\bar{0}}(A) \\ G_{-1}(A) &:= \left\langle x_{\tilde{\gamma}}(A) \mid \tilde{\gamma} \in \tilde{\Delta}_{-1} \right\rangle, & G_{-1,0}(A) &:= \left\langle h_H(A), x_{\tilde{\alpha}}(A) \mid H \in \mathfrak{h}_{\mathbb{Z}}, \tilde{\alpha} \in \tilde{\Delta}_{-1} \amalg \tilde{\Delta}_0 \right\rangle \end{aligned}$$

$$G_{\min}^-(A) := \left\langle x_{\tilde{\gamma}}(A), h_H(A), x_{\tilde{\alpha}}(A) \mid \tilde{\gamma} \in \tilde{\Delta}_{-1}, H \in \mathfrak{h}_{\mathbb{Z}}, \tilde{\alpha} \in \tilde{\Delta}_0^- \right\rangle$$

$$G_{\max}^+(A) := \left\langle h_H(A), x_{\tilde{\alpha}}(A), x_{\tilde{\beta}}(A) \mid H \in \mathfrak{h}_{\mathbb{Z}}, \tilde{\alpha} \in \tilde{\Delta}_0^+, \tilde{\beta} \in \tilde{\Delta}_z \forall z > 0 \right\rangle$$

(the last three rows only for $\mathfrak{g} \not\cong \tilde{S}(n)$). Let $G_{t\uparrow}, (G_{t\uparrow})_{\bar{0}}, G_{-1}, G_{-1,0}, G_{\min}^-, G_{\max}^+ : (\text{salg})_{\mathbb{k}} \longrightarrow$ (groups), for $t \geq -1$, be the corresponding full subgroup functors of G_V , and $\mathbf{G}_{t\uparrow}, (\mathbf{G}_{t\uparrow})_{\bar{0}}$ — for $t \geq -1$ — $\mathbf{G}_{-1}, \mathbf{G}_{-1,0}, \mathbf{G}_{\min}^-$ and \mathbf{G}_{\max}^+ the sheafification functors of the each of the above.

Lemma 4.13 and Lemma 4.8 yield the following properties of these special supersubgroups:

Proposition 4.34. *The following hold:*

- (a) $G_{t\uparrow} \cong \bigtimes_{\tilde{\alpha} \in \coprod_{z>t} \tilde{\Delta}_z} x_{\tilde{\gamma}} \cong \mathbf{G}_{t\uparrow}, (G_{t\uparrow})_{\bar{0}} \cong \bigtimes_{\tilde{\alpha} \in \coprod_{z>t} \tilde{\Delta}_z \cap \tilde{\Delta}_0} x_{\tilde{\gamma}} \cong (\mathbf{G}_{t\uparrow})_{\bar{0}}$ for all $t > -1$;
- (b) $G_{p\uparrow} = \mathbf{G}_{p\uparrow} \trianglelefteq \mathbf{G}_{q\uparrow} = G_{q\uparrow}, (G_{p\uparrow})_{\bar{0}} = (\mathbf{G}_{p\uparrow})_{\bar{0}} \trianglelefteq (\mathbf{G}_{q\uparrow})_{\bar{0}} = (G_{q\uparrow})_{\bar{0}},$ for all $-1 < q \leq p$;
- (c) $G_{-1,0} = G_{-1} \rtimes G_0, \mathbf{G}_{-1,0} = \mathbf{G}_{-1} \rtimes \mathbf{G}_0; G_{-1\uparrow} = G_0 \rtimes G_{0\uparrow}, \mathbf{G}_{-1\uparrow} = \mathbf{G}_0 \rtimes \mathbf{G}_{0\uparrow};$
- (d) $G_{\min}^{\pm} = G_{-1} \rtimes G_0^{\pm}, \mathbf{G}_{\min}^{\pm} = \mathbf{G}_{-1} \rtimes \mathbf{G}_0^{\pm},$ and $G_{\max}^{\pm} = G_0^{\pm} \rtimes G_{0\uparrow}, \mathbf{G}_{\max}^{\pm} = \mathbf{G}_0^{\pm} \rtimes \mathbf{G}_{0\uparrow}$ where G_0^{\pm} and \mathbf{G}_0^{\pm} are given as in Definition 4.5 and Definition 4.6 with respect to the splitting $\tilde{\Delta}_0 = \tilde{\Delta}_0^+ \coprod \tilde{\Delta}_0^-$ fixed in Definition 4.33.
- (e) the supergroup functors $\mathbf{G}_{t\uparrow}, (\mathbf{G}_{t\uparrow})_{\bar{0}}$ — for $t \geq -1$ — and — for $\mathfrak{g} \not\cong \tilde{S}(n)$ — $\mathbf{G}_{-1}, \mathbf{G}_{-1,0}, \mathbf{G}_{\min}^{\pm}$ and \mathbf{G}_{\max}^{\pm} are all representable, hence they are affine (algebraic) supergroups.

Finally, the “Lie’s Third Theorem” holds for these supersubgroups too, by the same arguments:

Theorem 4.35. *For every affine supergroup $\mathbf{G}_{t\uparrow}, (\mathbf{G}_{t\uparrow})_{\bar{0}}$ ($t \geq -1$), $\mathbf{G}_{-1}, \mathbf{G}_{-1,0}, \mathbf{G}_{\min}^{\pm}$ and \mathbf{G}_{\max}^{\pm} as above the corresponding tangent Lie algebra functor $\text{Lie}(-)$ is representable, namely*

$$\begin{aligned} \text{Lie}(\mathbf{G}_{t\uparrow}) &= \mathcal{L}_{\mathfrak{g}_{t\uparrow}^{\mathbb{Z}}} \quad , \quad \text{Lie}((\mathbf{G}_{t\uparrow})_{\bar{0}}) = \mathcal{L}_{(\mathfrak{g}_{t\uparrow}^{\mathbb{Z}})_{\bar{0}}} \quad \forall t \geq 0 \\ \text{Lie}(\mathbf{G}_{-1\uparrow}) &= \mathcal{L}_{(\mathfrak{g}_{-1\uparrow})_{V,\mathbb{k}}} \quad , \quad \text{Lie}((\mathbf{G}_{-1\uparrow})_{\bar{0}}) = \mathcal{L}_{(\mathfrak{g}_{-1\uparrow})_{V,\mathbb{k};\bar{0}}} \\ \text{Lie}(\mathbf{G}_{-1}) &= \mathcal{L}_{\mathfrak{g}_{-1}^{\mathbb{Z}}} \quad , \quad \text{Lie}(\mathbf{G}_{-1,0}) = \mathcal{L}_{(\mathfrak{g}_{-1,0})_{V,\mathbb{k}}} \\ \text{Lie}(\mathbf{G}_{\min}^{\pm}) &= \mathcal{L}_{(\mathfrak{b}_{\min}^{\pm})_{V,\mathbb{k}}} \quad , \quad \text{Lie}(\mathbf{G}_{\max}^{\pm}) = \mathcal{L}_{(\mathfrak{b}_{\max}^{\pm})_{V,\mathbb{k}}} \end{aligned}$$

as functors from $(\text{salg})_{\mathbb{k}}$ to $(\text{Lie})_{\mathbb{k}}$, where $(\mathfrak{g}_{-1\uparrow})_{V,\mathbb{k}} := (\mathfrak{g}_V)_0 \rtimes \mathfrak{g}_{0\uparrow}^{\mathbb{Z}}, (\mathfrak{g}_{-1\uparrow})_{V,\mathbb{k};\bar{0}} := (\mathfrak{g}_V)_0 \rtimes (\mathfrak{g}_{0\uparrow}^{\mathbb{Z}})_{\bar{0}},$ $(\mathfrak{g}_{-1,0})_{V,\mathbb{k}} := \mathfrak{g}_{-1}^{\mathbb{Z}} \rtimes (\mathfrak{g}_V)_0, (\mathfrak{b}_{\min}^{\pm})_{V,\mathbb{k}} := \mathfrak{g}_{-1}^{\mathbb{Z}} \rtimes (\mathfrak{g}_V)_0^{\pm}$ and $(\mathfrak{b}_{\max}^{\pm})_{V,\mathbb{k}} := (\mathfrak{g}_V)_0^{\pm} \rtimes \mathfrak{g}_{0\uparrow}^{\mathbb{Z}},$ with $(\mathfrak{g}_V)_0 = \mathfrak{h}_V \oplus (\bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_{\alpha}^{\mathbb{Z}})$ and $(\mathfrak{g}_V)_0^{\pm} = \mathfrak{h}_V \oplus (\bigoplus_{\alpha \in \Delta_0^{\pm}} \mathfrak{g}_{\alpha}^{\mathbb{Z}})$ — notation of Definition 2.23 and Definition 3.11.

4.8 The uniqueness theorem

In this subsection we shall prove that every connected affine algebraic \mathbb{k} -supergroup whose tangent Lie superalgebra is of Cartan type and whose *classical subgroup* (see below) is \mathbb{k} -split, is necessarily isomorphic to one of the supergroups \mathbf{G}_V we constructed. So, up to isomorphism the supergroups \mathbf{G}_V are the unique ones of the above mentioned type. We begin with a definition:

Definition 4.36. Let \mathbf{G} is an (affine) supergroup, $H := \mathcal{O}(\mathbf{G})$ the Hopf \mathbb{k} -superalgebra representing it, and $\overline{H} := H / (H_1^2 \oplus H_1) = H_0 / H_1^2$, which is a (classical) commutative Hopf algebra.

The affine group-scheme \mathbf{G}_{ev} represented by $\overline{H} = \overline{\mathcal{O}(\mathbf{G})}$ — so that $\mathcal{O}(\mathbf{G}_{ev}) = \overline{\mathcal{O}(\mathbf{G})}$ — is called the *classical supersubgroup(-scheme) associated with \mathbf{G}* . By construction, \mathbf{G}_{ev} coincides, as a group functor, with the restriction of \mathbf{G} to the category of commutative (unital, associative) \mathbb{k} -algebras.

The quotient map $\pi : H := \mathcal{O}(\mathbf{G}) \twoheadrightarrow \mathcal{O}(\mathbf{G}_{ev}) = \overline{H}$ yields an embedding $j : \mathbf{G}_{ev} \hookrightarrow \mathbf{G}$, so that \mathbf{G}_{ev} actually identifies with a closed (super)subgroup of \mathbf{G} .

Again by construction one has that every (closed) supersubgroup \mathbf{K} of \mathbf{G} which is *classical* is actually a (closed) subgroup of \mathbf{G}_{ev} . Moreover, the functor $Lie(\mathbf{G}_{ev})$ is the restriction of $Lie(\mathbf{G})$ to the category of (classical) commutative algebras: furthermore, when the latter is quasi-representable, say $Lie(\mathbf{G}) = \mathcal{L}_{\mathfrak{g}}$, then the former is quasi-representable too, with $Lie(\mathbf{G}_{ev}) = \mathcal{L}_{\mathfrak{g}_0}$.

Remark 4.37. Let $\mathbf{G} := \mathbf{G}_V$ be as in Definition 4.6. Then $(\mathbf{G}_V)_{ev} \cong \mathbf{G}_0$ and $(\mathbf{G}_V^\pm)_{ev} \cong \mathbf{G}_0^\pm$.

4.38. Supergroups with tangent Lie superalgebra of Cartan type. Let \mathbf{G} be a connected affine algebraic supergroup, defined over \mathbb{k} . We assume that the functor $Lie(\mathbf{G})$ associated with \mathbf{G} (cf. § 4.31) is quasi-representable, with $Lie(\mathbf{G}) = \mathcal{L}_{\mathfrak{g}_{V',\mathbb{k}}}$ where $\mathfrak{g}_{V',\mathbb{k}} := \mathbb{k} \otimes_{\mathbb{Z}} \mathfrak{g}_{V'}$: in particular, \mathfrak{g} is a simple Lie superalgebra of Cartan type, we fix in \mathfrak{g} a Chevalley basis, V' is a rational \mathfrak{g} -module with an admissible lattice, etc. In short, we might say that “ \mathbf{G} has tangent Lie superalgebra which is simple of Cartan type”. In particular, this means that $\mathfrak{g}_{V',\mathbb{k}}$ is free as a \mathbb{k} -module, with $rk_{\mathbb{k}}(\mathfrak{g}_{V',\mathbb{k}}) = \dim_{\mathbb{K}}(\mathfrak{g})$, and it is a Lie \mathbb{k} -superalgebra, whose \mathbb{Z}_2 -grading $\mathfrak{g}_{V',\mathbb{k}} = (\mathfrak{g}_{V',\mathbb{k}})_0 \oplus (\mathfrak{g}_{V',\mathbb{k}})_1$ is given by $(\mathfrak{g}_{V',\mathbb{k}})_{\bar{a}} := \mathbb{k} \otimes_{\mathbb{Z}} (\mathfrak{g}_{V'})_{\bar{a}}$ (for $a = 0, 1$) where $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the \mathbb{Z}_2 -grading of \mathfrak{g} . To simplify notation we shall drop all superscripts “ V', \mathbb{k} ”, writing just $\mathfrak{g}, \mathfrak{g}_0$ and \mathfrak{g}_1 , tacitly assuming that all these objects are \mathbb{k} -forms (specified above) of the initial objects defined over \mathbb{K} .

According to Definition 4.36, the supergroup \mathbf{G} has a “classical” subgroup \mathbf{G}_{ev} , such that $Lie(\mathbf{G}_{ev}) = \mathfrak{g}_0$. The assumptions also imply that \mathbf{G}_{ev} is a connected affine algebraic (classical) group-scheme, defined over \mathbb{k} . Now $\mathfrak{g}_0 = \mathfrak{g}_0 \oplus \mathfrak{g}_{0\uparrow}$ (cf. Definition 2.23), with \mathfrak{g}_0 reductive and $\mathfrak{g}_{0\uparrow}$ nilpotent: by the classical theory, from $Lie(\mathbf{G}_{ev}) = \mathfrak{g}_0$ we deduce that $\mathbf{G}_{ev} \cong \mathbf{G}'_0 \times \mathbf{G}'_{0\uparrow}$ for some connected algebraic groups \mathbf{G}'_0 and $\mathbf{G}'_{0\uparrow}$ such that $Lie(\mathbf{G}'_0) \cong \mathfrak{g}_0$ and $Lie(\mathbf{G}'_{0\uparrow}) \cong \mathfrak{g}_{0\uparrow}$. In particular, \mathbf{G}'_0 is reductive and $\mathbf{G}'_{0\uparrow}$ is unipotent. In addition, we assume that \mathbf{G} is \mathbb{k} -split, by which we mean — by definition — that the classical reductive group \mathbf{G}'_0 is \mathbb{k} -split.

In this subsection we show that \mathbf{G} is (isomorphic to) a “Cartan supergroup” \mathbf{G}_V associated with \mathfrak{g} and with some \mathfrak{g} -module V as in section 4.

For our arguments to apply, we need yet another technical requirement, namely we assume that \mathbf{G} is *linearizable*, i.e. it is embeddable into some $GL_{n|m}$ as a closed supersubgroup (this is true when $\mathbf{G} \cong \mathbf{G}_V$, hence it is a necessary condition). Note that this is automatically true when the ground ring \mathbb{k} is a field, or \mathbb{k} is a PID — e.g., $\mathbb{k} = \mathbb{Z}$ — and $\mathcal{O}(G)$ is free as a \mathbb{k} -module.

4.39. Linearizing \mathbf{G} . By classification theory of split reductive groups, \mathbf{G}'_0 can be realized via the classical Chevalley construction: namely, there is a faithful, rational, finite dimensional \mathfrak{g}_0 -module \widehat{V} , with an admissible lattice \widehat{M} , such that \mathbf{G}'_0 is isomorphic to the affine group-scheme associated with \mathfrak{g}_0 and with \widehat{V} by the classical Chevalley’s construction. Similarly, by classification theory of unipotent algebraic groups, $\mathbf{G}'_{0\uparrow}$ is isomorphic to the group $\mathbf{G}_{0\uparrow}$ in Definition 4.6. Overall, we get $\mathbf{G}_{ev} \cong \mathbf{G}'_0 \times \mathbf{G}_{0\uparrow}$. Actually, one has even more: $\mathbf{G}_{ev} \cong \mathbf{G}'_0 \times \mathbf{G}_{0\uparrow}$ can be realized at one strike by means of a (slight extension of the) classical Chevalley’s construction, based upon a faithful, rational, finite dimensional \mathfrak{g}_0 -module \widehat{V} with an admissible lattice \widehat{M} . Then the dual \mathfrak{g}_0 -module \widehat{V}^* is also faithful, rational, finite dimensional with \widehat{M}^* as an admissible lattice.

By assumption \mathbf{G} is *linearizable*, so it identifies with a closed supersubgroup of some $GL_{n|m}$. Then \mathbf{G}_{ev} identifies with a closed subgroup of $(GL_{n|m})_{ev}$, the classical subgroup of $GL_{n|m}$.

Pick the $(GL_{n|m})_{ev}$ -module $\widehat{U} := Ind_{\mathbf{G}_{ev}}^{(GL_{n|m})_{ev}}(\widehat{V}^*)$ — thought of as a functor from $(\text{salg})_{\mathbb{k}}$ to $(\mathbb{k}\text{-mod})$ — induced (by the classical theory of representations of algebraic groups) from the \mathbf{G}_{ev} -module \widehat{V}^* . Let \widehat{U}^* be the $(GL_{n|m})_{ev}$ -module dual to \widehat{U} ; as $Ind_{\mathbf{G}_{ev}}^{(GL_{n|m})_{ev}}(\widehat{V}^*)$ maps onto \widehat{V}^* , we have that $\widehat{V} \cong \widehat{V}^{**}$ embeds into \widehat{U}^* , i.e. \widehat{U}^* contains a \mathbf{G}_{ev} -submodule isomorphic to \widehat{V} .

Now, as $Lie((GL_{n|m})_{ev}) = (\mathfrak{gl}_{n|m})_0$, the $(GL_{n|m})_{ev}$ -module \widehat{U}^* is also a $(\mathfrak{gl}_{n|m})_0$ -module. As $(\mathfrak{gl}_{n|m})_0$ is a Lie (super)subalgebra of $\mathfrak{gl}_{n|m}$, we can perform on \widehat{U}^* the induction from $(\mathfrak{gl}_{n|m})_0$ to $\mathfrak{gl}_{n|m}$: this yields a $\mathfrak{gl}_{n|m}$ -module $W := Ind_{(\mathfrak{gl}_{n|m})_0}^{\mathfrak{gl}_{n|m}}(\widehat{U}^*)$, described by

$$W := Ind_{(\mathfrak{gl}_{n|m})_0}^{\mathfrak{gl}_{n|m}}(\widehat{U}^*) = U(\mathfrak{gl}_{n|m}) \otimes_{U((\mathfrak{gl}_{n|m})_0)} \widehat{U}^*$$

Now W is a $\mathrm{GL}_{n|m}$ -module too: indeed, one simply has to restrict the action of $\mathcal{L}_{\mathfrak{gl}_{n|m}}$ to $\mathrm{GL}_{n|m}$ (thought of as a subfunctor of $\mathcal{L}_{\mathfrak{gl}_{n|m}}$). Yet we need to describe the $\mathrm{GL}_{n|m}$ -action on W explicitly.

It is known that $\mathrm{GL}_{n|m}$ “splits” into a direct product — as a superscheme — of the subgroup $(\mathrm{GL}_{n|m})_{ev}$ and the totally odd supersubscheme $(\mathrm{GL}_{n|m})_{odd} := I + (\mathfrak{gl}_{n|m})_{\bar{1}}$, where $I := I_{n+m}$ is the identity (block) matrix of size $(n+m) \times (n+m)$: the splitting is given by the matrix product, namely $\mathrm{GL}_{n|m}(A) \cong (\mathrm{GL}_{n|m})_{ev}(A) \times (\mathrm{GL}_{n|m})_{odd}(A)$ via the unique factorization $\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} I_n & | & a^{-1}\beta \\ d^{-1}\gamma & | & I_m \end{pmatrix}$ for each block matrix $\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \in \mathrm{GL}_{n|m}(A)$, with $A \in (\mathrm{salg})_{\mathbb{k}}$.

Every $g_{ev} \in (\mathrm{GL}_{n|m})_{ev}(A)$ acts on any decomposable tensor $y \otimes \hat{u} \in U(\mathfrak{gl}_{n|m}) \otimes_{U((\mathfrak{gl}_{n|m})_{\bar{0}})} \hat{U}^* = W$ via $g_{ev} \cdot (y \otimes \hat{u}) = \mathrm{Ad}(g_{ev})(y) \otimes g_{ev} \cdot \hat{u}$, where on left-hand side we take the natural $\mathrm{GL}_{n|m}$ -action on $U(\mathfrak{gl}_{n|m})$ induced from the adjoint action on $\mathfrak{gl}_{n|m}$. Moreover, every $g_{odd} = I + g' \in (\mathrm{GL}_{n|m})_{odd}(A)$ acts on any $y \otimes \hat{u}$ as above by $g_{odd} \cdot (y \otimes \hat{u}) = (I + g') \cdot (y \otimes \hat{u}) = (y + g'y) \otimes \hat{u}$.

As \mathbf{G} is (embedded as) a closed supersubgroup of $\mathrm{GL}_{n|m}$, the $\mathrm{GL}_{n|m}$ -module W is also a \mathbf{G} -module. Moreover, by Remark 3.13, both for \mathbf{G} and for $\mathrm{GL}_{n|m}$ — which is a “Chevalley supergroup” in the sense of [9] — the Kostant superalgebra (with scalars extended to \mathbb{k}) identifies with the superalgebra of distributions: then Remark 3.13 tells also that $U_{\mathbb{k}}(\mathfrak{g})$ embeds into $U_{\mathbb{k}}(\mathfrak{gl}_{n|m})$. Then we can consider inside the \mathbf{G} -module W the subspace $V := U_{\mathbb{k}}(\mathfrak{g}) \otimes_{U_{\mathbb{k}}(\mathfrak{g}_{\bar{0}})} \hat{V}$: also, it is clear (thanks to the explicit description of the \mathbf{G} -action) that this V is a \mathbf{G} -submodule of W .

Tracking through the whole construction, as \hat{V} is rational and faithful as a \mathbf{G}_{ev} -module we see that V in turn is rational and faithful as a \mathbf{G} -module. Thus \mathbf{G} embeds as a closed supersubgroup inside $\mathrm{GL}(V)$, and \mathbf{G}_{ev} as a closed subgroup of \mathbf{G} . Also, as \hat{M} is an admissible lattice in the $\mathfrak{g}_{\bar{0}}$ -module \hat{V} , we see that $M := K_{\mathbb{k}}(\mathfrak{g}) \otimes_{K_{\mathbb{k}}(\mathfrak{g}_{\bar{0}})} \hat{M}$ is admissible in the \mathfrak{g} -module V (hereafter, we write $K_{\mathbb{k}}(\mathfrak{t}) := \mathbb{k} \otimes_{\mathbb{Z}} K_{\mathbb{Z}}(\mathfrak{t})$, and “admissible lattice” has the obvious meaning when passing from \mathbb{Z} -modules to \mathbb{k} -modules). Finally, as \hat{V} is finite dimensional, and $K_{\mathbb{k}}(\mathfrak{g})$ is (free) of finite rank as a $K_{\mathbb{k}}(\mathfrak{g}_{\bar{0}})$ -module (see Corollary 3.12) we argue that V is finite dimensional too.

By construction — including the fact that $V = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} \hat{V} = \bigwedge \mathfrak{g}_{\bar{1}} \otimes \hat{V}$ as a $\mathfrak{g}_{\bar{0}}$ -module is just $\hat{V}^{\oplus r}$ for $r := \mathrm{rank}_{U(\mathfrak{g}_{\bar{0}})}(U(\mathfrak{g})) = 2^{\dim(\mathfrak{g}_{\bar{1}})}$ — the $\mathfrak{g}_{\bar{0}}$ -action on V is just a diagonalization (r times) of the $\mathfrak{g}_{\bar{0}}$ -action on \hat{V} : as a consequence, the embedded copy of $(\mathbf{G}_V)_{ev}$ inside $\mathrm{GL}(V)$ is a (r times) diagonalized copy of the group obtained in $\mathrm{GL}(\hat{V})$ from the $\mathfrak{g}_{\bar{0}}$ -action on \hat{V} via the Chevalley construction. By assumption this group is \mathbf{G}_{ev} , thus $(\mathbf{G}_V)_{ev} = \mathbf{G}_{ev}$ inside $\mathrm{GL}(V)$.

4.40. Splitting \mathbf{G} . Recall that $\mathrm{GL}(V)$ splits as $\mathrm{GL}(V) = \mathrm{GL}(V)_{\bar{0}} \times \mathrm{GL}(V)_{\bar{1}}$, with $\mathrm{GL}(V)_{\bar{1}}(A) := I + \mathfrak{gl}(V)_{\bar{1}}(A_{\bar{1}})$ (cf. § 4.39). Then denote by $\pi_{\bar{0}}$ and $\pi_{\bar{1}}$ the projection maps of $\mathrm{GL}(V)$ onto $\mathrm{GL}(V)_{\bar{0}}$ and $\mathrm{GL}(V)_{\bar{1}}$. Note that $\mathrm{GL}(V)_{\bar{0}} = \pi_{\bar{0}}(\mathrm{GL}(V))$ coincides with $(\mathrm{GL}(V))_{ev} := \mathrm{GL}(V)|_{(\mathrm{alg})_{\mathbb{k}}}$.

Look at \mathbf{G} embedded inside $\mathrm{GL}(V)$: then $\pi_{\bar{0}}(\mathbf{G})$ is the restriction $\mathbf{G}|_{(\mathrm{alg})_{\mathbb{k}}}$, hence $\pi_{\bar{0}}(\mathbf{G}) = \mathbf{G}_{ev}$ (by Definition 4.36), so $\pi_{\bar{0}}(\mathbf{G}) \leq \mathbf{G}$. Given $g \in \mathbf{G}(A)$, for $A \in (\mathrm{salg})_{\mathbb{k}}$, it factors as $g = g_{\bar{0}} \cdot g_{\bar{1}}$ with $g_{\bar{0}} := \pi_{\bar{0}}(g) \in \mathrm{GL}(V)_{\bar{0}}$, $g_{\bar{1}} := \pi_{\bar{1}}(g) \in \mathrm{GL}(V)_{\bar{1}}$. Then $g_{\bar{0}} \in \mathbf{G}_{ev}(A) \leq \mathbf{G}(A)$ and $\pi_{\bar{1}}(g) =: g_{\bar{1}} = g_{\bar{0}}^{-1}g \in \mathbf{G}(A)$, so $\pi_{\bar{1}}(g) \in \mathbf{G}(A)$ too; it follows that $\pi_{\bar{1}}(\mathbf{G}) \subseteq \mathbf{G}$ (as a supersubscheme) too. The outcome is that the factorization $\mathrm{GL}(V) = \mathrm{GL}(V)_{\bar{0}} \times \mathrm{GL}(V)_{\bar{1}} = \pi_{\bar{0}}(\mathrm{GL}(V)) \times \pi_{\bar{1}}(\mathrm{GL}(V))$ of $\mathrm{GL}(V)$ induces the factorization $\mathbf{G} = \pi_{\bar{0}}(\mathbf{G}) \times \pi_{\bar{1}}(\mathbf{G}) = \mathbf{G}_{ev} \times \pi_{\bar{1}}(\mathbf{G})$ of \mathbf{G} as well.

4.41. Construction of a supergroup \mathbf{G}_V and comparison with \mathbf{G} . The \mathbf{G} -module V constructed in § 4.39 is obviously also a representation of the Lie superalgebra $\mathrm{Lie}(\mathbf{G}) = \mathfrak{g}$. More precisely, $V = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_{\bar{0}})} \hat{V}$ implies $V = \mathrm{Ind}_{\mathfrak{g}_{\bar{0}}}^{\mathfrak{g}}(\hat{V})$. In addition, we saw that: (a) V has finite dimension, (b) V is rational, (c) V contains $M := K_{\mathbb{k}}(\mathfrak{g}) \otimes_{K_{\mathbb{k}}(\mathfrak{g}_{\bar{0}})} \hat{M}$ as an admissible lattice.

Therefore, using V and M we can construct the affine algebraic supergroup \mathbf{G}_V , as in section 4, which is embedded inside $\mathrm{GL}(V)$ as a closed (cf. Proposition 4.26) connected supersubgroup.

By the analysis above, we can embed both \mathbf{G} and \mathbf{G}_V as closed supersubgroups of $\mathrm{GL}(V)$: thus we identify \mathbf{G} and \mathbf{G}_V with their images in $\mathrm{GL}(V)$, and their tangent Lie superalgebras with the corresponding images in $\mathfrak{gl}(V)$. We can now prove the main result of this subsection:

Theorem 4.42. *Let \mathbf{G} and \mathbf{G}_V be as above. Then $\mathbf{G} = \mathbf{G}_V$.*

Proof. By the analysis in §4.40, the splitting $\mathrm{GL}(V) = \mathrm{GL}(V)_{\bar{0}} \times \mathrm{GL}(V)_{\bar{1}}$, with $\mathrm{GL}(V)_{\bar{1}}(A) := I + \mathfrak{gl}(V)_{\bar{1}}(A_{\bar{1}})$, and the embedding of \mathbf{G} in $\mathrm{GL}(V)$ provide a splitting $\mathbf{G} = \mathbf{G}_{ev} \times \mathbf{G}_{odd}$, with $\mathbf{G}_{ev} = \pi_{\bar{0}}(\mathbf{G})$ and $\mathbf{G}_{odd} := \pi_{\bar{1}}(\mathbf{G})$. Similarly, the same fact with \mathbf{G}_V replacing \mathbf{G} yields $\mathbf{G}_V = (\mathbf{G}_V)_{ev} \times (\mathbf{G}_V)_{odd}$, with $(\mathbf{G}_V)_{ev} = \pi_{\bar{0}}(\mathbf{G}_V) = \mathbf{G}_{\bar{0}}$ and $(\mathbf{G}_V)_{odd} := \pi_{\bar{1}}(\mathbf{G}_V)$ — see Remark 4.37. All these splittings are superscheme isomorphisms given by the group product map. So as $\mathbf{G} = \mathbf{G}_{ev} \cdot \mathbf{G}_{odd}$ and $\mathbf{G}_V = (\mathbf{G}_V)_{ev} \cdot (\mathbf{G}_V)_{odd}$, it is enough to prove $\mathbf{G}_{ev} = (\mathbf{G}_V)_{ev}$ and $\mathbf{G}_{odd} = (\mathbf{G}_V)_{odd}$.

First, the identity $\mathbf{G}_{ev} = (\mathbf{G}_V)_{ev}$ follows from §4.39. Indeed, therein we last pointed out that (the copy of) \mathbf{G}_{ev} inside $\mathrm{GL}(V)$ can be realized through the classical Chevalley's construction via the $\mathfrak{g}_{\bar{0}}$ -module V and the lattice M ; but this is exactly the same outcome as first performing the construction of the supergroup \mathbf{G}_V and then taking its classical subgroup $(\mathbf{G}_V)_{ev}$, so we are done.

Second, definitions yield $\mathbf{G}_{odd} = I + T_I(\mathbf{G}_{odd})$ as a supersubscheme of $\mathrm{GL}(V)_{\bar{1}}$, where $T_I(\mathbf{G}_{odd})$ is the tangent superspace to \mathbf{G}_{odd} at I ; similarly $(\mathbf{G}_V)_{odd} = I + T_I((\mathbf{G}_V)_{odd})$. But by construction we have also $T_I(\mathbf{G}_{odd}) = \mathfrak{g}_{\bar{1}} = T_I((\mathbf{G}_V)_{odd})$, hence $\mathbf{G}_{odd} = (\mathbf{G}_V)_{odd}$. \square

5 The standard case

In this section we look somewhat in detail the example of the supergroup $\mathbf{G}_{\Lambda(n)}$ associated with $\mathfrak{g} := W(n)$ and with the “standard” \mathfrak{g} -module $V := \Lambda(n)$ — i.e., the defining representation of $\mathfrak{g} := W(n)$. Our analysis then can be easily adapted to the case $\mathfrak{g} := S(n)$ and $V := \Lambda(n)$ again. More in general, as each Cartan type Lie superalgebra is naturally embedded in $W(n)$, from the present analysis one can also deduce (with some extra work) a similar analysis for the other cases.

5.1 The affine algebraic supergroup $\mathbf{G}_{\Lambda(n)}$

From now on, we retain the notation of subsection 2.3, and we let $\mathfrak{g} := W(n) = \mathrm{Der}_{\mathbb{K}}(\Lambda(n))$ and $V := \Lambda(n)$. Fix the \mathbb{K} -bases $B_{\Lambda(n)} := \{ \underline{\xi}^{\underline{e}} \mid \underline{e} \in \{0, 1\}^n \}$ in $\Lambda(n)$ and $B_{W(n)} := \{ \underline{\xi}^{\underline{a}} \partial_i \mid \underline{a} \in \{0, 1\}^n, i = 1, \dots, n \}$ in $W(n)$ — see subsections 2.3, 2.10 and 3.3. Recall that $B_{W(n)}$ is a Chevalley basis of $\mathfrak{g} := W(n)$: the root vectors $X_{\tilde{\alpha}} \ (\tilde{\alpha} \in \tilde{\Delta})$ are the $\underline{\xi}^{\underline{a}} \partial_i$ with $\underline{a} \neq \underline{e}_i$, while the “toral type” elements H_i are just the remaining elements $\xi_i \partial_i$ of $B_{W(n)}$ ($i = 1, \dots, n$).

By definition, \mathfrak{g} acts on $V := \Lambda(n)$ by (super)derivations. Explicitly, the action of any basis element in $B_{W(n)}$ onto any basis element in $B_{\Lambda(n)}$ reads

$$\underline{\xi}^{\underline{a}} \partial_i (\underline{\xi}^{\underline{e}}) = \pm \delta_{1, \underline{e}(i)} \underline{\xi}^{\underline{a} + \underline{e} - \underline{e}_i} \quad (5.1)$$

This simple formula has deep consequences. The first is that *the $W(n)$ -module $\Lambda(n)$ is rational*, as the H_i act diagonally with integral eigenvalues. A second consequence is that

$$(\underline{\xi}^{\underline{a}} \partial_i)^2 = 0 \quad \forall \underline{a} \neq \underline{e}_i, \quad (\xi_i \partial_i)^2 = \xi_i \partial_i \quad \forall i = 1, \dots, n \quad (5.2)$$

(note that $\underline{\xi}^{\underline{e}_i} = \xi_i$). The left-hand part of (5.2) implies that all divided powers $X_{\tilde{\alpha}}^{(m)}$ of even root vectors ($\tilde{\alpha} \in \tilde{\Delta}$) with $m > 1$ act as zero on $\Lambda(n)$. From this and from (5.1) it follows at once that *the \mathbb{Z} -span of $B_{\Lambda(n)}$, call it M , is an admissible lattice of Λ* .

As another consequence, we can describe the one-parameter supersubgroups $x_{\tilde{\alpha}}$ and h_i associated with root vectors $X_{\tilde{\alpha}} = \underline{\xi}^{\underline{a}} \partial_i$ ($\underline{a} \neq \underline{e}_i$) and “toral” elements $H_i = \xi_i \partial_i$. For $x_{\tilde{\alpha}}$ one has $x_{\tilde{\alpha}}(\mathbf{u}) := \exp(\mathbf{u} X_{\tilde{\alpha}}) = \sum_{m=0}^{+\infty} (\mathbf{u} X_{\tilde{\alpha}})^m / m! = 1 + \mathbf{u} X_{\tilde{\alpha}} = 1 + \mathbf{u} \underline{\xi}^{\underline{a}} \partial_i$, for any $A \in (\mathrm{salg})_{\mathbb{K}}$ and $\mathbf{u} \in A_{p(\tilde{\alpha})}$, where $p(\tilde{\alpha})$ is the parity of $\pi(\tilde{\alpha})$. Matching this with (5.1), the action of $x_{\tilde{\alpha}}(\mathbf{u})$ on basis elements of $\Lambda(n)(A) := A_{\bar{0}} \otimes_{\mathbb{Z}} M_{\bar{0}} + A_{\bar{1}} \otimes_{\mathbb{Z}} M_{\bar{1}}$ reads (for $\mathbf{t} \in A_{p(\underline{e})}$)

$$\mathbf{t} \underline{\xi}^{\underline{e}} \xrightarrow{x_{\tilde{\alpha}}(\mathbf{u})} x_{\tilde{\alpha}}(\mathbf{u})(\mathbf{t} \underline{\xi}^{\underline{e}}) = \mathbf{t} \underline{\xi}^{\underline{e}} + \mathbf{u} \underline{\xi}^{\underline{a}} \partial_i (\mathbf{t} \underline{\xi}^{\underline{e}}) = \mathbf{t} \underline{\xi}^{\underline{e}} \pm \delta_{1, \underline{e}(i)} \mathbf{u} \mathbf{t} \underline{\xi}^{\underline{a} + \underline{e} - \underline{e}_i}$$

Similarly, for h_i we find $\mathbf{t} \underline{\xi}^{\varepsilon} \xrightarrow{h_i(u)} h_i(u)(\mathbf{t} \underline{\xi}^{\varepsilon}) = u^{\delta_{1,\varepsilon(i)}} \mathbf{t} \underline{\xi}^{\varepsilon}$ for all $\mathbf{t} \in A_{p(\underline{\varepsilon})}$.

In particular, this yields the following “point-set description” of $\mathbf{G}_{\Lambda(n)}$:

Proposition 5.1. *Let $\mathbf{G}_{\Lambda(n)}$ be the supergroup associated with $\mathfrak{g} := W(n)$ and the $W(n)$ -module $\Lambda(n)$ as in Section 4, and let us fix total orders in $\tilde{\Delta}_{\bar{0}\uparrow}$ and in $\tilde{\Delta}_{\bar{1}}$ such that $\tilde{\Delta}_{\bar{1}}^- \preceq \tilde{\Delta}_{\bar{1}}^+$ or $\tilde{\Delta}_{\bar{1}}^+ \preceq \tilde{\Delta}_{\bar{1}}^-$. Then for any $A \in (\text{salg})_{\mathbb{k}}$ the group $\mathbf{G}_{\Lambda(n)}(A)$ is given by*

$$\mathbf{G}_{\Lambda(n)}(A) = \text{GL}_n(A_{\bar{0}}) \times \left(\times_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}} (1 + A_{\bar{0}} X_{\tilde{\alpha}}) \right) \times \left(\times_{\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}} (1 + A_{\bar{1}} X_{\tilde{\gamma}}) \right)$$

(the products indexed by $\tilde{\Delta}_{\bar{0}\uparrow}$ or by $\tilde{\Delta}_{\bar{1}}$ being ordered according to the fixed orders), as well as by all set-theoretic factorizations that one gets by permuting the three factors above with one another.

Proof. As we noticed in §4.27, $\mathbf{G}_{\Lambda(n)}$ factors into $\mathbf{G}_{\Lambda(n)} \cong \mathbf{G}_0 \times \mathbf{G}_{\bar{0}\uparrow} \times \mathbf{G}_{\bar{1}}^{\leq}$, and in addition $\mathbf{G}_{\bar{1}}^{\leq} \cong \times_{\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}} x_{\tilde{\gamma}}(A)$, $\mathbf{G}_{\bar{0}\uparrow} \cong \times_{\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}} x_{\tilde{\alpha}}(A)$ and $\mathbf{G}_0 \cong \mathbf{Ch}_{\Lambda(n)}$. The latter is the standard (affine, algebraic) group functor associated by the classical Chevalley construction with $\mathfrak{g}_0 \cong \mathfrak{gl}_n$ and with the \mathfrak{gl}_n -module $\Lambda(n)$: but the very construction clearly gives $\mathbf{G}_0 \cong \mathbf{Ch}_{\Lambda(n)} \cong \text{GL}_n$. Finally, taking into account the additional remark that $x_{\tilde{\alpha}}(A) = (1 + A_{\bar{0}} X_{\tilde{\alpha}})$ for $\tilde{\alpha} \in \tilde{\Delta}_{\bar{0}\uparrow}$ and $x_{\tilde{\gamma}}(A) = (1 + A_{\bar{1}} X_{\tilde{\gamma}})$ for $\tilde{\gamma} \in \tilde{\Delta}_{\bar{1}}$ — by the above analysis — we end up with the claim. \square

Remark 5.2. The factorization of $\mathbf{G}_{\Lambda(n)}(A)$ in the above Proposition is a special instance of the general result in §4.27. But the present case is much easier to handle, as commutation relations among one-parameter supersubgroups (as in Lemma 4.8) look simpler: e.g., for $\tilde{\alpha}, \tilde{\beta} \in \tilde{\Delta}_{\bar{0}\uparrow} \amalg \tilde{\Delta}_{\bar{1}}$ one has $(x_{\tilde{\alpha}}(\mathbf{p}), x_{\tilde{\beta}}(\mathbf{q})) = (1 + \mathbf{p} X_{\tilde{\alpha}}, 1 + \mathbf{q} X_{\tilde{\beta}}) = 1 + \mathbf{p} \mathbf{q} [X_{\tilde{\alpha}}, X_{\tilde{\beta}}]$ where (cf. Examples 3.3(a)) the bracket $[X_{\tilde{\alpha}}, X_{\tilde{\beta}}]$ is either zero, or a root vector, or a sum (with signs) of two such vectors.

5.2 $\mathbf{G}_{\Lambda(n)}$ as a supergroup of automorphisms

In the present subsection we prove that the supergroup functor $\mathbf{G}_{\Lambda(n)}$ actually is a group functor of automorphisms, namely the group functor of superalgebra automorphisms canonically associated with the \mathbb{k} -superalgebra $\Lambda(n)$. We begin with a (general) definition.

Definition 5.3. Let $\mathfrak{A} \in (\text{salg})_{\mathbb{k}}$ be any \mathbb{k} -superalgebra which, as a \mathbb{k} -module, is free of finite rank. We define the supergroup functor $\mathbf{Aut}(\mathfrak{A}) : (\text{salg})_{\mathbb{k}} \rightarrow (\text{groups})$ as the full subfunctor of the group functor $\text{GL}(\mathfrak{A}_{\bullet})$ — cf. 2.6(b) — whose value on objects is $\mathbf{Aut}(\mathfrak{A})(A) := \text{Aut}_{(\text{salg})_A}(\mathfrak{A}_A)$ — the group of all A -linear superalgebra automorphisms of $\mathfrak{A}_A := A \otimes_{\mathbb{k}} \mathfrak{A}$ — for all $A \in (\text{salg})_{\mathbb{k}}$.

5.4. The group functor $\mathbf{Aut}(\Lambda(n))$. Given the \mathbb{k} -superalgebra $\mathfrak{A} = \Lambda(n)$, we are interested into $\mathbf{Aut}(\Lambda(n))$: our ultimate goal is to show that $\mathbf{G}_{\Lambda(n)} = \mathbf{Aut}(\Lambda(n))$. Note that

$$\mathbf{Aut}(\Lambda(n))(A) = \text{Aut}_{(\text{salg})_A}(\Lambda(n)_A) = \text{Aut}_{(\text{salg})_A}(A[\xi_1, \dots, \xi_n]) \quad \forall A \in (\text{salg})_{\mathbb{k}} \quad (5.3)$$

because $\Lambda(n)_A := A \otimes_{\mathbb{k}} \Lambda(n) = A \otimes_{\mathbb{k}} \mathbb{k}[\xi_1, \dots, \xi_n] = A[\xi_1, \dots, \xi_n]$. Now, given $A \in (\text{salg})_{\mathbb{k}}$, any $\phi \in \text{Aut}_{(\text{salg})_A}(A[\xi_1, \dots, \xi_n])$ is uniquely determined by the images of the ξ_j : these are of the form

$$\phi(\xi_j) = \kappa_j + \sum_{i=1}^n c_{i,j} \xi_i + \sum_{\substack{|\underline{e}| > 1 \\ |\underline{e}| \text{ is even}}} \kappa_{\underline{e},j} \underline{\xi}^{\underline{e}} + \sum_{\substack{|\underline{e}| > 1 \\ |\underline{e}| \text{ is odd}}} c_{\underline{e},j} \underline{\xi}^{\underline{e}} \quad \forall j = 1, \dots, n \quad (5.4)$$

with the only constraints that each $\phi(\xi_j)$ be again *odd*, which means $\kappa_j, \kappa_{\underline{e},j} \in A_{\bar{1}}$, $c_{i,j}, c_{\underline{e},j} \in A_{\bar{0}}$, and that ϕ itself be invertible. By the nilpotency of the ξ_t , it is clear that ϕ is invertible if and only if the square matrix of the $c_{i,j}$'s is invertible, i.e. $C_{\phi} := (c_{i,j})_{i=1, \dots, n; j=1, \dots, n}^{j=1, \dots, n; i=1, \dots, n} \in \text{GL}_n(A_{\bar{0}})$. Note also that (5.4) means that $\phi \in \text{Aut}_{(\text{salg})_A}(\Lambda(n)_A) = \text{Aut}_{(\text{salg})_A}(A[\xi_1, \dots, \xi_n])$ can be written as

$$\phi = \sum_{j=1}^n \kappa_j \partial_j + \sum_{i=1}^n \sum_{j=1}^n c_{i,j} \xi_i \partial_j + \sum_{j=1}^n \sum_{\substack{|\underline{e}| > 1 \\ |\underline{e}| \text{ is even}}} \kappa_{\underline{e},j} \underline{\xi}^{\underline{e}} \partial_j + \sum_{j=1}^n \sum_{\substack{|\underline{e}| > 1 \\ |\underline{e}| \text{ is odd}}} c_{\underline{e},j} \underline{\xi}^{\underline{e}} \partial_j \quad (5.5)$$

Thus every $\phi \in \text{Aut}_{(\text{salg})_A}(\Lambda(n)_A)$ is uniquely associated with a string of coefficients: the κ_j , the $\kappa_{\underline{e},j}$, the $c_{i,j}$ and the $c_{\underline{e},j}$ as above. Therefore, the overall conclusion is the following:

Proposition 5.5. *The group functor $\mathbf{Aut}(\Lambda(n))$ is representable — hence it is an (affine) supergroup scheme — and isomorphic, as a superscheme, to $\mathbb{A}^{0|n} \times \mathrm{GL}_n \times \mathbb{A}^{|\tilde{\Delta}_{0^\dagger}|} \times \mathbb{A}^{0|\tilde{\Delta}_{1^\dagger}|}$.*

We are now ready for the main result of this subsection:

Theorem 5.6. $\mathbf{G}_{\Lambda(n)} = \mathbf{Aut}(\Lambda(n))$, that is $\mathbf{G}_{\Lambda(n)}$ coincides with the group functor $\mathbf{Aut}(\Lambda(n))$.

Proof. By construction, we must prove that $\mathbf{G}_{\Lambda(n)}(A) = \mathbf{Aut}(\Lambda(n))(A) := \mathrm{Aut}_{(\mathrm{salg})_A}(\Lambda(n)_A)$ with $\Lambda(n)_A := A \otimes_{\mathbb{k}} \Lambda(n) = A[\xi_1, \dots, \xi_n]$, $A \in (\mathrm{salg})_{\mathbb{k}}$. We begin by $\mathbf{G}_{\Lambda(n)}(A) \subseteq \mathrm{Aut}_{(\mathrm{salg})_A}(\Lambda(n)_A)$.

By Remarks 4.7(d), $\mathbf{G}_{\Lambda(n)}(A)$ is a subgroup of $\mathrm{GL}(V_A)$: we must only prove that its elements are superalgebra automorphisms. As $\mathbf{G}_{\Lambda(n)}$ is the sheafification of $G_{\Lambda(n)}$, and \mathbf{Aut} is a sheaf, it is enough to prove that $G_{\Lambda(n)}(A) \subseteq \mathrm{Aut}_{(\mathrm{salg})_A}(\Lambda(n)_A)$. Now, the group $G_{\Lambda(n)}(A)$ is generated by such elements as $x_{\tilde{\alpha}}(t) := \exp(t X_{\tilde{\alpha}})$, $x_{\tilde{\beta}}(\vartheta) := \exp(\vartheta X_{\tilde{\beta}})$, $h_i(u) := u^{H_i}$; both $X_{\tilde{\alpha}}$ and $X_{\tilde{\beta}}$ are superderivations of $\Lambda(n)$, hence they also define (uniquely) A -linear superderivations of $\Lambda(n)_A := A \otimes_{\mathbb{k}} \Lambda(n)$. But then both $t X_{\tilde{\alpha}}$ and $\vartheta X_{\tilde{\beta}}$ are A -linear derivations of $\Lambda(n)_A$ into itself: taking their exponentials we get (A -linear) automorphisms of $\Lambda(n)_A$, so that $x_{\tilde{\alpha}}(t), x_{\tilde{\beta}}(\vartheta) \in \mathrm{Aut}_{(\mathrm{salg})_A}(\Lambda(n)_A)$. A similar argument proves $h_i(u) \in \mathrm{Aut}_{(\mathrm{salg})_A}(\Lambda(n)_A)$, hence we are done.

Now we prove that the above inclusion is an identity. We begin with an aside observation: by the explicit description of automorphisms in (5.5), one sees that for each $A \in (\mathrm{salg})_{\mathbb{k}}$ the subsets

$$\begin{aligned} \mathbf{Aut}(\Lambda(n))_{\leq 0}(A) &:= \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_{\underline{e},j} = 0 = c_{\underline{e},j} \ \forall \underline{e}, j \} \\ \mathbf{Aut}(\Lambda(n))_{0^\dagger}(A) &:= \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = 0 = c_{i,j} \ \forall i, j \} \end{aligned}$$

are subgroups of $\mathbf{Aut}(\Lambda(n))(A)$, which altogether generate $\mathbf{Aut}(\Lambda(n))(A)$. This defines two supersubgroups $\mathbf{Aut}(\Lambda(n))_{\leq 0}$ and $\mathbf{Aut}(\Lambda(n))_{0^\dagger}$ which jointly generate $\mathbf{Aut}(\Lambda(n))$.

The first supersubgroup $\mathbf{Aut}(\Lambda(n))_{\leq 0}$ is isomorphic to the algebraic group $\mathbf{Aff}_n = \mathbb{G}_a^{\times n} \rtimes \mathrm{GL}_n$ of all affine-linear transformations of the (totally odd) affine superspace $\mathbb{A}^{0|n}$. It contains the subgroup $G'_{\leq 0}(A)$ generated by all the elements $(1 + \vartheta \partial_j) = x_{-\varepsilon_j}(\vartheta)$, $(1 + t \xi_i \partial_j) = x_{\varepsilon_i - \varepsilon_j}(t)$ and $h_i(u)$ of $\mathbf{G}_{\Lambda(n)}(A)$ — for $\vartheta \in A_{\bar{1}}$, $t \in A_{\bar{0}}$, $u \in U(A_{\bar{0}})$, $i, j = 1, \dots, n$. All these $G'_{\leq 0}(A)$ define a supergroup functor, whose sheafification $\mathbf{G}'_{\leq 0}$ clearly coincides with $\mathbf{Aut}(\Lambda(n))_{\leq 0}$.

The second supersubgroup $\mathbf{Aut}(\Lambda(n))_{0^\dagger}$ contains the subgroup $G'_{0^\dagger}(A)$ generated by the elements $(1 + \mathbf{t} \xi_{\underline{e}} \partial_j) = x_{\alpha_{\underline{e},j}}(\mathbf{t})$ — for $\mathbf{t} \in A_{\bar{0}} \cup A_{\bar{1}}$, $i, j = 1, \dots, n$, where $\alpha_{\underline{e},j}$ is the unique element of $\tilde{\Delta}$ associated with \underline{e} and j . We shall now show that $G'_{0^\dagger}(A)$ coincides with $\mathbf{Aut}(\Lambda(n))_{0^\dagger}$: by the previous analysis, this will be enough to prove that $\mathbf{Aut}(\Lambda(n)) = \mathbf{G}_V$. Consider the subsets

$$\mathbf{Aut}(\Lambda(n))_{>t}(A) := \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = 0 = c_{i,j}, \kappa_{\underline{e},j} = 0 = c_{\underline{e},j} \ \forall i, j, \forall |\underline{e}| \leq t \}$$

for all $t = 2, \dots, n$; then easy computations, basing upon (5.5) and upon the formula

$$(1 + \mathbf{t}' \xi_{\underline{a}} \partial_j) (1 + \mathbf{t}'' \xi_{\underline{b}} \partial_k) = 1 + \mathbf{t}' \xi_{\underline{a}} \partial_j + \mathbf{t}'' \xi_{\underline{b}} \partial_k \pm \delta_{\underline{b}(j),1} \mathbf{t}' \mathbf{t}'' \xi_{\underline{a}+\underline{b}-\varepsilon_j} \partial_k$$

show that these subsets form a strictly decreasing sequence of normal subgroups $\mathbf{Aut}(\Lambda(n))_{0^\dagger}$, which ends with the trivial subgroup. It is immediate to see that $\mathbf{Aut}(\Lambda(n))_{0^\dagger} / \mathbf{Aut}(\Lambda(n))_{>2}(A)$ is generated by the cosets $(1 + \mathbf{t} \xi_{\underline{e}} \partial_j) \bmod \mathbf{Aut}(\Lambda(n))_{>2}(A)$ with $|\underline{e}| = 2$. Similarly, one sees easily for all t (by iteration) that $\mathbf{Aut}(\Lambda(n))_{0^\dagger} / \mathbf{Aut}(\Lambda(n))_{>t}(A)$ is generated by the cosets $(1 + \mathbf{t} \xi_{\underline{e}} \partial_j) \bmod \mathbf{Aut}(\Lambda(n))_{>t}(A)$ with $2 \leq |\underline{e}| \leq t$. For $t = n$ this yields the expected result. \square

5.3 Special supersubgroups of $\mathbf{G}_{\Lambda(n)}$

We finish this section with an explicit description of the special supersubgroups of $\mathbf{G}_{\Lambda(n)}$ that we considered along the way — cf. §4.3 and §4.7 — namely (for all $t \geq -1$)

$$\mathbf{G}_{-1}, \quad \mathbf{G}_0, \quad \mathbf{G}_{-1,0}, \quad \mathbf{G}_{\bar{0}}, \quad \mathbf{G}_{0^\dagger}, \quad \mathbf{G}_{t^\dagger}, \quad (\mathbf{G}_{t^\dagger})_{\bar{0}}, \quad \mathbf{G}_{\min}^\pm, \quad \mathbf{G}_{\max}^\pm$$

Such a description follows from Propositions 4.10, 4.11, 4.12, 4.14 and 4.34; using the identification $\mathbf{G}_{\Lambda(n)} = \mathbf{Aut}(\Lambda(n))$ in Theorem 5.6 and by (5.5), all those results yield easily the following:

Proposition 5.7. *For every $A \in (\text{salg})_{\mathbb{k}}$, we have:*

(a) $\mathbf{G}_{-1}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid c_{i,j} = \kappa_{\underline{e},j} = c_{\underline{e},j} = 0 \ \forall \ \underline{e}, i, j \} \cong \mathbb{G}_{a,odd}^{\times n}(A)$, so that $\mathbf{G}_{-1} \cong \mathbb{G}_{a,odd}^{\times n}$, where $\mathbb{G}_{a,odd}$ is defined on objects by $A \mapsto \mathbb{G}_{a,odd}(A) := A_{\bar{1}}$ (as additive group);

(b) $\mathbf{G}_0(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = \kappa_{\underline{e},j} = c_{\underline{e},j} = 0 \ \forall \ \underline{e}, j \} \cong \text{GL}_n(A)$, so that $\mathbf{G}_0 \cong \text{GL}_n$, where GL_n is the classical general linear affine group extended to superalgebras via $(\text{salg})_{\mathbb{k}} \ni A \mapsto \text{GL}_n(A) := \text{GL}_n(A_{\bar{0}})$;

(c) $\mathbf{G}_{-1,0}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_{\underline{e},j} = c_{\underline{e},j} = 0 \ \forall \ \underline{e}, j \} =: \mathbf{Aut}(\Lambda(n))_{\leq 0}(A)$, so that $\mathbf{G}_{-1,0} = \mathbf{G}_{-1} \rtimes \mathbf{G}_0 \cong \mathbb{G}_{a,odd}^{\times n} \rtimes \text{GL}_n =: \mathbf{Aff}_{0|n}$, the latter being the (classical) algebraic group of all affine-linear transformations of the totally odd affine superspace $\mathbb{A}^{0|n}$;

(d) $\mathbf{G}_{0\uparrow}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = c_{i,j} = \kappa_{\underline{e},j} = 0 \ \forall \ \underline{e}, i, j \}$, so that $\mathbf{G}_{0\uparrow} \cong \mathbb{A}_{\mathbb{k}}^{N_{0\uparrow}|0}$ as affine superschemes, where $N_{0\uparrow} := |\tilde{\Delta}_{0\uparrow}| = \sum_{z>0} |\tilde{\Delta}_z \cap \tilde{\Delta}_{\bar{0}}|$;

(e) $\mathbf{G}_{\bar{0}}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = \kappa_{\underline{e},j} = 0 \ \forall \ \underline{e}, j \}$, so that $\mathbf{G}_{\bar{0}} = \mathbf{G}_0 \rtimes \mathbf{G}_{0\uparrow} \cong \text{GL}_n \times \mathbb{A}_{\mathbb{k}}^{N_{0\uparrow}|0}$ as affine superschemes;

(f) $\mathbf{G}_{t\uparrow}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = c_{i,j} = \kappa_{\underline{e},j} = c_{\underline{e}',j} = 0 \ \forall \ i, j, \underline{e}, \forall \ \underline{e}': |\underline{e}'| \leq t+1 \}$ for all $t > -1$, therefore $\mathbf{G}_{t\uparrow} \cong \mathbb{A}_{\mathbb{k}}^{N_{t\uparrow}^{\bar{0}}|N_{t\uparrow}^{\bar{1}}}$ as affine superschemes, where $N_{t\uparrow}^{\bar{s}} := \sum_{z>t} |\tilde{\Delta}_z \cap \tilde{\Delta}_{\bar{s}}|$; in particular, $\mathbf{G}_{0\uparrow}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = c_{i,j} = 0 \ \forall \ i, j \}$, hence $\mathbf{G}_{0\uparrow} \cong \mathbb{A}_{\mathbb{k}}^{N_{0\uparrow}^+|N_{0\uparrow}^-}$;

(g) $\mathbf{G}_{-1\uparrow}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = 0 \ \forall \ j \}$, so $\mathbf{G}_{-1\uparrow} = \mathbf{G}_0 \rtimes \mathbf{G}_{0\uparrow} \cong \text{GL}_n \rtimes \mathbb{A}_{\mathbb{k}}^{N_{0\uparrow}^+|N_{0\uparrow}^-}$ as affine superschemes;

(h) $(\mathbf{G}_{t\uparrow})_{\bar{0}}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = c_{i,j} = \kappa_{\underline{e},j} = c_{\underline{e}',j} = 0 \ \forall \ i, j, \underline{e}, \forall \ \underline{e}': |\underline{e}'| \leq t+1 \}$ for all $t > -1$, therefore $(\mathbf{G}_{t\uparrow})_{\bar{0}} \cong \mathbb{A}_{\mathbb{k}}^{N_{t\uparrow}^+|0}$ as (totally even) affine superschemes; in particular, $(\mathbf{G}_{0\uparrow})_{\bar{0}}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = c_{i,j} = \kappa_{\underline{e},j} = 0 \ \forall \ i, j, \underline{e} \}$, hence $\mathbf{G}_{0\uparrow} \cong \mathbb{A}_{\mathbb{k}}^{N_{0\uparrow}^+|0}$;

(i) $(\mathbf{G}_{-1\uparrow})_{\bar{0}}(A) = \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = \kappa_{\underline{e},j} = 0 \ \forall \ i, j, \underline{e} \}$, thus $\mathbf{G}_{-1\uparrow} = \mathbf{G}_0 \rtimes \mathbf{G}_{0\uparrow} \cong \text{GL}_n \rtimes \mathbb{A}_{\mathbb{k}}^{N_{0\uparrow}^+|0}$ as (totally even) affine superschemes;

(j) let $\tilde{\Delta}_0 = \tilde{\Delta}_0^+ \amalg \tilde{\Delta}_0^-$ be the splitting of the root system $\tilde{\Delta}_0 = \Delta_0$ of $\mathfrak{g}_0 = \mathfrak{gl}_n$ given by $\tilde{\Delta}_0^+ := \{ \varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n \}$ and $\tilde{\Delta}_0^- := \{ \varepsilon_i - \varepsilon_j \mid 1 \leq j < i \leq n \}$, and define \mathbf{G}_{\min}^- and \mathbf{G}_{\max}^+ accordingly as in Definition 4.33. Then we have

$$\begin{aligned} \mathbf{G}_{\min}^-(A) &= \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid c_{i,j} = \kappa_{\underline{e},j} = c_{\underline{e},j} = 0 \ \forall \ i, j, \underline{e} : i < j \} \cong \mathbf{G}_{-1} \rtimes \mathbf{B}^- \\ \mathbf{G}_{\max}^+(A) &= \{ \phi \in \mathbf{Aut}(\Lambda(n))(A) \mid \kappa_j = c_{i,j} = 0 \ \forall \ i, j : i > j \} \cong \mathbf{B}^+ \rtimes \mathbf{G}_{0\uparrow} \end{aligned}$$

where \mathbf{B}^{\pm} is the Borel subgroup of $\mathbf{G}_0 = \text{GL}_n$ of all invertible upper/lower triangular matrices.

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